Howework 4 – Introduction to Computational Science

Claudio Maggioni

May 17, 2020

Question 1

$$L_0(x) = \prod_{j=0, j\neq 0}^{n} \frac{x - x_j}{x_i - x_j} = \frac{x-(-0.5)}{(-1)-(-0.5)} \cdot \frac{x-0.5}{(-1)-0.5} \cdot \frac{x-1}{(-1)-1} = -\frac{2}{3}x^3 + \frac{2}{3}x^2 + \frac{1}{6}x - \frac{1}{6}$$

$$L_1(x) = \prod_{j=0, j\neq 1}^{n} \frac{x - x_j}{x_i - x_j} = \frac{x-(-1)}{(-0.5)-(-1)} \cdot \frac{x-0.5}{(-0.5)-0.5} \cdot \frac{x-1}{(-0.5)-1} = \frac{4}{3}x^3 - \frac{2}{3}x^2 - \frac{4}{3}x + \frac{2}{3}$$

$$L_2(x) = \prod_{j=0, j\neq 2}^{n} \frac{x - x_j}{x_i - x_j} = \frac{x-(-1)}{0.5-(-1)} \cdot \frac{x-(-0.5)}{0.5-(-0.5)} \cdot \frac{x-1}{0.5-1} = -\frac{4}{3}x^3 - \frac{2}{3}x^2 + \frac{4}{3}x + \frac{2}{3}$$

$$L_3(x) = \prod_{j=0, j\neq 3}^{n} \frac{x - x_j}{x_i - x_j} = \frac{x-(-1)}{1-(-1)} \cdot \frac{x-(-0.5)}{1-(-0.5)} \cdot \frac{x-0.5}{1-0.5} = \frac{2}{3}x^3 + \frac{2}{3}x^2 - \frac{1}{6}x - \frac{1}{6}$$

$$p(x) = \sum_{i=0}^{n} y_i L_i(x) = 2 \cdot \left(-\frac{2}{3}x^3 + \frac{2}{3}x^2 + \frac{1}{6}x - \frac{1}{6}\right) + 1 \cdot \left(\frac{4}{3}x^3 - \frac{2}{3}x^2 - \frac{4}{3}x + \frac{2}{3}\right) +$$

$$0.5 \cdot \left(-\frac{4}{3}x^3 - \frac{2}{3}x^2 + \frac{4}{3}x + \frac{2}{3}\right) + 0.4 \cdot \left(\frac{2}{3}x^3 + \frac{2}{3}x^2 - \frac{1}{6}x - \frac{1}{6}\right) = -\frac{2}{5}x^3 + \frac{3}{5}x^2 - \frac{2}{5}x + \frac{3}{5}$$

$$\frac{\max_{x\in[-1,1]} |f^{(n+1)}|}{4!} = \frac{\max_{x\in[-1,1]} |\frac{768}{|2x+3|^6}|}{24} = \max_{x\in[-1,1]} \frac{32}{|2x+3|^6} = 32$$

$$\max_{x\in[-1,1]} \left|(x-1)\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right)(x+1)\right| = \max_{x\in[-1,1]} \left|x^4 - \frac{5}{4}x^2 + \frac{1}{4}\right| = \frac{1}{4}$$

$$\max_{x\in[-1,1]} |f(x) - p(x)| \leq \frac{\max_{x\in[-1,1]} |f^{(n+1)}|}{4!} \max_{x\in[-1,1]} \left|(x-1)\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right)(x+1)\right| =$$

$$= 32 \cdot \frac{1}{4} = 8 \leq 8$$

The statement above is true so p satisfies the error estimate:

$$\max_{x\in[-1,1]} |f(x) - p(x)| \leq 8$$

Question 2

We first use the Lagrange method:

$$L_1(x) = \prod_{j=0,j\neq 1}^{2} \frac{x-x_j}{x_i-x_j} = \frac{x-0}{1-0}\frac{x-3}{1-3} = -\frac{1}{2}x^2 + \frac{3}{2}x$$

$$L_2(x) = \prod_{j=0,j\neq 2}^{2} \frac{x-x_j}{x_i-x_j} = \frac{x-0}{3-0}\frac{x-1}{3-1} = \frac{1}{6}x^2 - \frac{1}{6}x$$

$$p(x) = (-3)\cdot\left(-\frac{1}{2}x^2 + \frac{3}{2}x\right) + 1\cdot\left(\frac{1}{6}x^2 - \frac{1}{6}x\right) = \frac{5}{3}x^2 - \frac{14}{3}x$$

Then we use the Newtonian method:

$$a_0 = f[0] = 0, \qquad f[1] = -3 \qquad f[3] = 1$$

$$a_1 = f[0,1] = \frac{-3-0}{1-0} = -3, \qquad f[1,3] = \frac{1-(-3)}{3-1} = 2$$

$$a_2 = f[0,1,3] = \frac{2-(-3)}{3-0} = \frac{5}{3}$$

$$p(x) = \left(\frac{5}{3}(x-1) - 3\right)x + 0 = \frac{5}{3}x^2 - \frac{14}{3}x$$

The interpolating polynomials are indeed equal.
Now we use the Horner method to compute $p(0.5)$:

$$y = a_n = a_2 = \frac{5}{3}$$

$$i = 1$$

$$y = y(x-x_1) + a_1 = \frac{5}{3}(0.5-1) + a_1 - 3 = -\frac{5}{6} - 3 = -\frac{23}{6}$$

$$i = 0$$

$$y = y(x-x_0) + a_0 = -\frac{23}{6}\cdot 0.5 + 0 = -\frac{23}{12}$$

Question 4

Point a)

The node coordinates to which fix the quadratic spline are $(1/2, y_0), (3/2, y_1), (5/2, y_2), (7/2, y_3)$.
Then, we can start formulating the equations for the linear system:

$$y_0 = s\left(\frac{1}{2}\right) = a_{-1}B_2\left(\frac{1}{2}+1\right) + a_0B_2\left(\frac{1}{2}\right) + a_1B_2\left(\frac{1}{2}-1\right) +$$

$$+ a_2B_2\left(\frac{1}{2}-2\right) + a_{-1}B_2\left(\frac{1}{2}-3\right) + a_0B_2\left(\frac{1}{2}-4\right) + a_1B_2\left(\frac{1}{2}-5\right) =$$

$$a_{-1}\cdot 0 + a_0\cdot\frac{1}{2} + a_1\cdot\frac{1}{2} + a_2\cdot 0 + a_{-1}\cdot 0 + a_0\cdot 0 + a_1\cdot 0 = \frac{a_0+a_1}{2}$$

$$y_1 = s\left(\frac{3}{2}\right) = a_{-1}B_2\left(\frac{3}{2}+1\right) + a_0B_2\left(\frac{3}{2}\right) + a_1B_2\left(\frac{3}{2}-1\right) +$$

$$+a_2B_2\left(\frac{3}{2}-2\right) + a_{-1}B_2\left(\frac{3}{2}-3\right) + a_0B_2\left(\frac{3}{2}-4\right) + a_1B_2\left(\frac{3}{2}-5\right) =$$

$$a_{-1}\cdot 0 + a_0\cdot 0 + a_1\cdot\frac{1}{2} + a_2\cdot\frac{1}{2} + a_{-1}\cdot 0 + a_0\cdot 0 + a_1\cdot 0 = \frac{a_1+a_2}{2}$$

$$y_2 = s\left(\frac{5}{2}\right) = a_{-1}B_2\left(\frac{5}{2}+1\right) + a_0B_2\left(\frac{5}{2}\right) + a_1B_2\left(\frac{5}{2}-1\right) +$$

$$+a_2B_2\left(\frac{5}{2}-2\right) + a_{-1}B_2\left(\frac{5}{2}-3\right) + a_0B_2\left(\frac{5}{2}-4\right) + a_1B_2\left(\frac{5}{2}-5\right) =$$

$$a_{-1}\cdot 0 + a_0\cdot 0 + a_1\cdot 0 + a_2\cdot\frac{1}{2} + a_{-1}\cdot\frac{1}{2} + a_0\cdot 0 + a_1\cdot 0 = \frac{a_2+a_{-1}}{2}$$

$$y_1 = s\left(\frac{7}{2}\right) = a_{-1}B_2\left(\frac{7}{2}+1\right) + a_0B_2\left(\frac{7}{2}\right) + a_1B_2\left(\frac{7}{2}-1\right) +$$

$$+a_2B_2\left(\frac{7}{2}-2\right) + a_{-1}B_2\left(\frac{7}{2}-3\right) + a_0B_2\left(\frac{7}{2}-4\right) + a_1B_2\left(\frac{7}{2}-5\right) =$$

$$a_{-1}\cdot 0 + a_0\cdot 0 + a_1\cdot 0 + a_2\cdot 0 + a_{-1}\cdot\frac{1}{2} + a_0\cdot\frac{1}{2} + a_1\cdot 0 = \frac{a_{-1}+a_0}{2}$$

The linear system in matrix form is:

$$\frac{1}{2}\cdot\begin{bmatrix}0&1&1&0\\0&0&1&1\\1&0&0&1\\1&1&0&0\end{bmatrix}\begin{bmatrix}a_{-1}\\a_0\\a_1\\a_2\end{bmatrix} = \begin{bmatrix}y_0\\y_1\\y_2\\y_3\end{bmatrix}$$

The determinant of that matrix is 0 so the matrix is singular.

Point b)

The node coordinates to which fix the quadratic spline are $(0, y_0), (1, y_1), (2, y_2), (3, y_3)$.

Then, we can start formulating the equations for the linear system:

$$y_0 = s(0) = a_{-1}B_2(1) + a_0B_2(0) + a_1B_2(-1) +$$

$$+a_2B_2(-2) + a_{-1}B_2(-3) + a_0B_2(-4) + a_1B_2(-5) =$$

$$a_{-1}\cdot\frac{1}{8} + a_0\cdot\frac{3}{4} + a_1\cdot\frac{1}{8} + a_2\cdot 0 + a_{-1}\cdot 0 + a_0\cdot 0 + a_1\cdot 0 = \frac{1}{8}a_{-1} + \frac{3}{4}a_0 + \frac{1}{8}a_1$$

$$y_1 = s(1) = a_{-1}B_2(1+1) + a_0B_2(1) + a_1B_2(1-1) +$$

$$+a_2B_2(1-2) + a_{-1}B_2(1-3) + a_0B_2(1-4) + a_1B_2(1-5) =$$

$$a_{-1}\cdot 0 + a_0\cdot\left(\frac{1}{8}\right) + a_1\cdot\frac{3}{4} + a_2\cdot\left(\frac{1}{8}\right) + a_{-1}\cdot 0 + a_0\cdot 0 + a_1\cdot 0 = \frac{1}{8}a_0 + \frac{3}{4}a_1 + \frac{1}{8}a_2$$

$$y_2 = s(2) = a_{-1}B_2(2+1) + a_0B_2(2) + a_1B_2(2-1) +$$

$$+a_2B_2(2-2) + a_{-1}B_2(2-3) + a_0B_2(2-4) + a_1B_2(2-5) =$$

$$a_{-1} \cdot 0 + a_0 \cdot 0 + a_1 \cdot \left(\frac{1}{8}\right) + a_2 \cdot \frac{3}{4} + a_{-1} \cdot \left(\frac{1}{8}\right) + a_0 \cdot 0 + a_1 \cdot 0 = \frac{1}{8}a_1 + \frac{3}{4}a_2 + \frac{1}{8}a_{-1}$$

$$y_3 = s(3) = a_{-1}B_2(3+1) + a_0B_2(3) + a_1B_2(3-1) +$$
$$+a_2B_2(3-2) + a_{-1}B_2(3-3) + a_0B_2(3-4) + a_1B_2(3-5) =$$
$$a_{-1} \cdot 0 + a_0 \cdot 0 + a_1 \cdot 0 + a_2 \cdot \frac{1}{8} + a_{-1} \cdot \frac{3}{4} + a_0 \cdot \frac{1}{8} + a_1 \cdot 0 = \frac{1}{8}a_2 + \frac{3}{4}a_{-1} + \frac{1}{8}a_0$$

The linear system in matrix form is:

$$\frac{1}{8} \cdot \begin{bmatrix} 6 & 1 & 0 & 1 \\ 1 & 6 & 1 & 0 \\ 0 & 1 & 6 & 1 \\ 1 & 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_{-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Question 5

$$1 = y_0 = s(x_0) = 0 = a_{-1}B_3(1) + a_0B_3(0) + a_1B_3(-1) + a_2B_3(-2) + a_3B_3(-3) = \frac{1}{6}a_{-1} + \frac{2}{3}a_0 + \frac{1}{6}a_1$$

$$5 = y_1 = s(x_1) = 1 = a_{-1}B_3(2) + a_0B_3(1) + a_1B_3(0) + a_2B_3(-1) + a_3B_3(-2) = \frac{1}{6}a_0 + \frac{2}{3}a_1 + \frac{1}{6}a_2$$

$$1 = y_2 = s(x_2) = 2 = a_{-1}B_3(3) + a_0B_3(2) + a_1B_3(1) + a_2B_3(0) + a_3B_3(-1) = \frac{1}{6}a_1 + \frac{2}{3}a_2 + \frac{1}{6}a_3$$

$$s''(0) = a_{-1}B_3''(1) + a_0B_3''(0) + a_1B_3''(-1) + a_2B_3''(-2) + a_3B_3''(-3) = a_{-1} - 2a_0 + a_1$$

$$s''(2) = a_{-1}B_3''(3) + a_0B_3''(2) + a_1B_3''(1) + a_2B_3''(-0) + a_3B_3''(-1) = a_1 - 2a_2 + a_3$$

$$\frac{1}{6} \cdot \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} a_{-1} \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ y_0 \\ y_1 \\ y_2 \\ 0 \end{bmatrix}$$

We then use Gaussian *ellimination* to solve the system.

$$\begin{array}{ccccc|c} 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 6 \\ 0 & 1 & 4 & 1 & 0 & 30 \\ 0 & 0 & 1 & 4 & 1 & 6 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array} \qquad \begin{array}{ccccc|c} 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 6 \\ 0 & 1 & 4 & 1 & 0 & 30 \\ 0 & 0 & 1 & 4 & 1 & 6 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array} \qquad \begin{array}{ccccc|c} 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 30 \\ 0 & 6 & 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 4 & 1 & 6 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}$$

$$\begin{array}{ccccc|c} 1 & 0 & 9 & 2 & 0 & 60 \\ 0 & 1 & 4 & 1 & 0 & 30 \\ 0 & 0 & -24 & -6 & 0 & -174 \\ 0 & 0 & 1 & 4 & 1 & 6 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array} \qquad \begin{array}{ccccc|c} 1 & 0 & 9 & 2 & 0 & 60 \\ 0 & 1 & 4 & 1 & 0 & 30 \\ 0 & 0 & 1 & 4 & 1 & 6 \\ 0 & 0 & -24 & -6 & 0 & -174 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array} \qquad \begin{array}{ccccc|c} 1 & 0 & 0 & -34 & -9 & 6 \\ 0 & 1 & 0 & -15 & -4 & 6 \\ 0 & 0 & 1 & 4 & 1 & 6 \\ 0 & 0 & 0 & 90 & 24 & -30 \\ 0 & 0 & 0 & 6 & 0 & -6 \end{array}$$

$$
\begin{array}{ccccc|c}
1 & 0 & 0 & -34 & -9 & 6 \\
0 & 1 & 0 & -15 & -4 & 6 \\
0 & 0 & 1 & 4 & 1 & 6 \\
0 & 0 & 0 & 1 & 4/15 & -1/3 \\
0 & 0 & 0 & 6 & 0 & -6
\end{array}
\qquad
\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & 1/15 & -16/3 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & -1/15 & 22/3 \\
0 & 0 & 0 & 1 & 4/15 & -1/3 \\
0 & 0 & 0 & 0 & 8/5 & -8
\end{array}
\qquad
\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & 0 & -5 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 7 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & -5
\end{array}
$$

Therefore, the coefficients are $a_{-1} = a_3 = -5$, $a_0 = a_2 = 1$, and $a_1 = 7$.