Howework 5 – Introduction to Computational Science

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Question 1

Given the definition of degree of exactness being the highest polynomial degree n at which a quadrature, for every polynomial of degree n, produces exactly the same polynomial, these are the proofs.

Midpoint rule

All polynomials of degree 1 can be expressed as:

$$p_1(x) = a_1 \cdot x + a_0$$

Therefore their integral is:

$$\int_0^1 a_1 \cdot x + a_0 dx = \frac{a_1}{2} + a_0$$

The midpoint rule for $p_1(x)$ is

$$f\left(\frac{1}{2}\right) \cdot 1 = \frac{a_1}{2} + a_0 = \int_0^1 a_1 \cdot x + a_0 dx$$

Therefore the midpoint rule has a degree of exactness of at least 1.

It is easy to show that the degree of exactness is not higher than 1 by considering the degree 2 polynomial x^2, which has an integral in $[0, 1]$ of $\frac{1}{3}$ but a midpoint rule quadrature of $\frac{1}{4}$.

Trapezoidal rule

The proof is similar to the one for the midpoint rule, but with this quadrature for degree 1 polynomials:

$$\frac{f(0)}{2} + \frac{f(1)}{2} = \frac{a_0 + a_1 + a_0}{2} = \frac{a_1}{2} + a_0$$

Which is again equal to the general integral for these polynomials.

Again x^2 is a degree 2 polynomial with integral $\frac{1}{3}$ but a midpoint quadrature of $\frac{0+1}{2} = \frac{1}{2}$, thus bounding the degree of exactness to 1.

Simpson rule

The proof is again similar, but for degree 3 polynomials which can all be written as:

$$p_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

The integral is:

$$\int_0^1 p_3(x)dx = \frac{a_3}{4} + \frac{a_2}{3} + \frac{a_1}{2} + a_0$$

The Simpson rule gives:

$$\frac{1}{6} \cdot f(0) + \frac{4}{6} \cdot f\left(\frac{1}{2}\right) + \frac{1}{6} \cdot f(1) = \frac{1}{6}a_0 + \frac{4}{6}\left(\frac{a_3}{8} + \frac{a_2}{4} + \frac{a_1}{2} + a_0\right) +$$

$$\frac{1}{6}(a_3 + a_2 + a_1 + a_0) = \frac{a_3}{4} + \frac{a_2}{3} + \frac{a_1}{2} + a_0 = \int_0^1 p_3(x)dx$$

Which tells us that the degree of exactness is at least 1.

We can bound the degree of exactness to 3 with the 4th degree polynomial x^4 which has integral in $[0, 1]$ of $\frac{1}{5}$ but has a quadrature of $\frac{1}{6} \cdot 0 + \frac{2}{3} \cdot \frac{1}{16} + \frac{1}{6} \cdot 1 = \frac{5}{24}$.

Question 2

The algebraic solution is:

$$\int_0^1 1 - 4(x - 0.5)^2 dx = 1 - 4 \cdot \int_0^1 x^2 - x + \frac{1}{4} = 1 - 4\left(\frac{4 - 6 + 3}{12}\right) = 1 - \frac{1}{3} = \frac{2}{3}$$

The solution using quadrature is:

$$Q = \frac{1}{2}(f(0) + f(1)) = \frac{1}{2}(0 + 0) = 0 \qquad (x, h) = \left(\frac{1}{2}, \frac{1}{2}\right) \qquad \epsilon = \frac{1}{10}$$

$$E\left(\frac{1}{2}, \frac{1}{2}\right) = f\left(\frac{1}{2}\right) - \frac{1}{2}(f(0) + f(1)) = 1 > \frac{1}{10} \qquad Q = 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$$

$$E\left(\frac{1}{4}, \frac{1}{4}\right) = f\left(\frac{1}{4}\right) - \frac{1}{2}\left(f(0) + f\left(\frac{1}{2}\right)\right) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4} > \frac{1}{10} \qquad Q = \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{4} = \frac{9}{16}$$

$$E\left(\frac{3}{4}, \frac{1}{4}\right) = f\left(\frac{3}{4}\right) - \frac{1}{2}\left(f\left(\frac{1}{2}\right) + f(1)\right) = \frac{1}{4} > \frac{1}{10} \qquad Q = \frac{9}{16} + \frac{1}{4} \cdot \frac{1}{4} = \frac{10}{16}$$

$$E\left(\frac{1}{8}, \frac{1}{8}\right) = f\left(\frac{1}{8}\right) - \frac{1}{2}\left(f(0) + f\left(\frac{1}{4}\right)\right) = \frac{7}{16} - \frac{1}{2} \cdot \frac{3}{4} = \frac{1}{16} < \frac{1}{10}$$

$$E\left(\frac{3}{8}, \frac{1}{8}\right) = f\left(\frac{3}{8}\right) - \frac{1}{2}\left(f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right)\right) = \frac{15}{16} - \frac{1}{2} \cdot \frac{7}{4} = \frac{1}{16} < \frac{1}{10}$$

$$E\left(\frac{5}{8}, \frac{1}{8}\right) = f\left(\frac{5}{8}\right) - \frac{1}{2}\left(f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\right) = \frac{15}{16} - \frac{1}{2} \cdot \frac{7}{4} = \frac{1}{16} < \frac{1}{10}$$

$$E\left(\frac{7}{8}, \frac{1}{8}\right) = f\left(\frac{7}{8}\right) - \frac{1}{2}\left(f\left(\frac{3}{4}\right) + f(1)\right) = \frac{7}{16} - \frac{1}{2} \cdot \frac{3}{4} = \frac{1}{16} < \frac{1}{10}$$

Thus the solution using quadrature is $\frac{5}{8}$.

Question 4

$$\begin{bmatrix} x_0^2 & x_0 & 1 \\ x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0 & x_1 & x_2 & x_3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_0^2 & x_0 & 1 \\ x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0 & x_1 & x_2 & x_3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} 18 & 8 & 6 \\ 8 & 6 & 2 \\ 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

We now use Gaussian *ellimination* to solve the system:

$$\begin{array}{ccc|c} 18 & 8 & 6 & 2 \\ 8 & 6 & 2 & 0 \\ 6 & 2 & 4 & 2 \end{array} \qquad \begin{array}{ccc|c} 1 & \frac{4}{9} & \frac{1}{3} & \frac{1}{9} \\ 8 & 6 & 2 & 0 \\ 6 & 2 & 4 & 2 \end{array} \qquad \begin{array}{ccc|c} 1 & \frac{4}{9} & \frac{1}{3} & \frac{1}{9} \\ 0 & \frac{22}{9} & \frac{-2}{3} & \frac{-8}{9} \\ 0 & \frac{-2}{3} & 2 & \frac{4}{3} \end{array} \qquad \begin{array}{ccc|c} 1 & \frac{4}{9} & \frac{1}{3} & \frac{1}{9} \\ 0 & 1 & \frac{-3}{11} & \frac{-4}{11} \\ 0 & \frac{-2}{3} & 2 & \frac{4}{3} \end{array}$$

$$\begin{array}{ccc|c} 1 & 0 & \frac{5}{11} & \frac{3}{11} \\ 0 & 1 & \frac{-3}{11} & \frac{-4}{11} \\ 0 & 0 & \frac{20}{11} & \frac{12}{11} \end{array} \qquad \begin{array}{ccc|c} 1 & 0 & \frac{5}{11} & \frac{3}{11} \\ 0 & 1 & \frac{-3}{11} & \frac{-4}{11} \\ 0 & 0 & 1 & \frac{3}{5} \end{array} \qquad \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{-1}{5} \\ 0 & 0 & 1 & \frac{3}{5} \end{array} \qquad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{5} \\ \frac{3}{5} \end{bmatrix}$$