

Howework 4 – Introduction to Computational Science

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Question 1

$$L_0(x) = \prod_{j=0, j \neq 0}^n \frac{x - x_j}{x_i - x_j} = \frac{x - (-0.5)}{(-1) - (-0.5)} \cdot \frac{x - 0.5}{(-1) - 0.5} \cdot \frac{x - 1}{(-1) - 1} = -\frac{2}{3}x^3 + \frac{2}{3}x^2 + \frac{1}{6}x - \frac{1}{6}$$

$$L_1(x) = \prod_{j=0, j \neq 1}^n \frac{x - x_j}{x_i - x_j} = \frac{x - (-1)}{(-0.5) - (-1)} \cdot \frac{x - 0.5}{(-0.5) - 0.5} \cdot \frac{x - 1}{(-0.5) - 1} = \frac{4}{3}x^3 - \frac{2}{3}x^2 - \frac{4}{3}x + \frac{2}{3}$$

$$L_2(x) = \prod_{j=0, j \neq 2}^n \frac{x - x_j}{x_i - x_j} = \frac{x - (-1)}{0.5 - (-1)} \cdot \frac{x - (-0.5)}{0.5 - (-0.5)} \cdot \frac{x - 1}{0.5 - 1} = -\frac{4}{3}x^3 - \frac{2}{3}x^2 + \frac{4}{3}x + \frac{2}{3}$$

$$L_3(x) = \prod_{j=0, j \neq 3}^n \frac{x - x_j}{x_i - x_j} = \frac{x - (-1)}{1 - (-1)} \cdot \frac{x - (-0.5)}{1 - (-0.5)} \cdot \frac{x - 0.5}{1 - 0.5} = \frac{2}{3}x^3 + \frac{2}{3}x^2 - \frac{1}{6}x - \frac{1}{6}$$

$$p(x) = \sum_{i=0}^n y_i L_i(x) = 2 \cdot \left(-\frac{2}{3}x^3 + \frac{2}{3}x^2 + \frac{1}{6}x - \frac{1}{6} \right) + 1 \cdot \left(\frac{4}{3}x^3 - \frac{2}{3}x^2 - \frac{4}{3}x + \frac{2}{3} \right) + 0.5 \cdot \left(-\frac{4}{3}x^3 - \frac{2}{3}x^2 + \frac{4}{3}x + \frac{2}{3} \right) + 0.4 \cdot \left(\frac{2}{3}x^3 + \frac{2}{3}x^2 - \frac{1}{6}x - \frac{1}{6} \right) = -\frac{2}{5}x^3 + \frac{3}{5}x^2 - \frac{2}{5}x + \frac{3}{5}$$

$$\frac{\max_{x \in [-1,1]} |f^{(n+1)}|}{4!} = \frac{\max_{x \in [-1,1]} \left| \frac{768}{|2x+3|^6} \right|}{24} = \max_{x \in [-1,1]} \frac{32}{|2x+3|^6} = 32$$

$$\max_{x \in [-1,1]} \left| (x-1) \left(x - \frac{1}{2} \right) \left(x + \frac{1}{2} \right) (x+1) \right| = \max_{x \in [-1,1]} \left| x^4 - \frac{5}{4}x^2 + \frac{1}{4} \right| = \frac{1}{4}$$

$$\begin{aligned} \max_{x \in [-1,1]} |f(x) - p(x)| &\leq \frac{\max_{x \in [-1,1]} |f^{(n+1)}|}{4!} \max_{x \in [-1,1]} \left| (x-1) \left(x - \frac{1}{2} \right) \left(x + \frac{1}{2} \right) (x+1) \right| \\ &= 32 \cdot \frac{1}{4} = 8 \leq 8 \end{aligned}$$

The statement above is true so p satisfies the error estimate:

$$\max_{x \in [-1,1]} |f(x) - p(x)| \leq 8$$

Question 2

We first use the Lagrange method:

$$L_1(x) = \prod_{j=0, j \neq 1}^2 \frac{x - x_j}{x_i - x_j} = \frac{x - 0}{1 - 0} \frac{x - 3}{1 - 3} = -\frac{1}{2}x^2 + \frac{3}{2}x$$

$$L_2(x) = \prod_{j=0, j \neq 2}^2 \frac{x - x_j}{x_i - x_j} = \frac{x - 0}{3 - 0} \frac{x - 1}{3 - 1} = \frac{1}{6}x^2 - \frac{1}{6}x$$

$$p(x) = (-3) \cdot \left(-\frac{1}{2}x^2 + \frac{3}{2}x\right) + 1 \cdot \left(\frac{1}{6}x^2 - \frac{1}{6}x\right) = \frac{5}{3}x^2 - \frac{14}{3}x$$

Then we use the Newtonian method:

$$\begin{aligned} a_0 &= f[0] = 0, & f[1] &= -3 & f[3] &= 1 \\ a_1 &= f[0, 1] = \frac{-3 - 0}{1 - 0} = -3, & f[1, 3] &= \frac{1 - (-3)}{3 - 1} = 2 \\ a_2 &= f[0, 1, 3] = \frac{2 - (-3)}{3 - 0} = \frac{5}{3} \end{aligned}$$

$$p(x) = \left(\frac{5}{3}(x - 1) - 3\right)x + 0 = \frac{5}{3}x^2 - \frac{14}{3}x$$

The interpolating polynomials are indeed equal.

Now we use the Horner method to compute $p(0.5)$:

$$y = a_n = a_2 = \frac{5}{3}$$

$$i = 1$$

$$y = y(x - x_1) + a_1 = \frac{5}{3}(0.5 - 1) + a_1 - 3 = -\frac{5}{6} - 3 = -\frac{23}{6}$$

$$i = 0$$

$$y = y(x - x_0) + a_0 = -\frac{23}{6} \cdot 0.5 + 0 = -\frac{23}{12}$$

Question 4

Point a)

The node coordinates to which fix the quadratic spline are $(1/2, y_0), (1, y_1), (3/2, y_2), (2, y_3)$.

Then, we can start formulating the equations for the linear system:

$$\begin{aligned} y_0 &= s\left(\frac{1}{2}\right) = a_{-1}B_2\left(\frac{1}{2} + 1\right) + a_0B_2\left(\frac{1}{2}\right) + a_1B_2\left(\frac{1}{2} - 1\right) + \\ &+ a_2B_2\left(\frac{1}{2} - 2\right) + a_{-1}B_2\left(\frac{1}{2} - 3\right) + a_0B_2\left(\frac{1}{2} - 4\right) + a_1B_2\left(\frac{1}{2} - 5\right) = \\ &a_{-1} \cdot 0 + a_0 \cdot \frac{1}{2} + a_1 \cdot \frac{1}{2} + a_2 \cdot 0 + a_{-1} \cdot 0 + a_0 \cdot 0 + a_1 \cdot 0 = \frac{a_0 + a_1}{2} \end{aligned}$$

$$\begin{aligned}
y_1 = s(1) &= a_{-1}B_2(1+1) + a_0B_2(1) + a_1B_2(1-1) + \\
&+ a_2B_2(1-2) + a_{-1}B_2(1-3) + a_0B_2(1-4) + a_1B_2(1-5) = \\
a_{-1} \cdot 0 + a_0 \cdot \left(\frac{1}{8}\right) + a_1 \cdot \frac{3}{4} + a_2 \cdot \left(\frac{1}{8}\right) + a_{-1} \cdot 0 + a_0 \cdot 0 + a_1 \cdot 0 &= \frac{1}{8}a_0 + \frac{3}{4}a_1 + \frac{1}{8}a_2
\end{aligned}$$

$$\begin{aligned}
y_2 = s\left(\frac{3}{2}\right) &= a_{-1}B_2\left(\frac{3}{2}+1\right) + a_0B_2\left(\frac{3}{2}\right) + a_1B_2\left(\frac{3}{2}-1\right) + \\
&+ a_2B_2\left(\frac{3}{2}-2\right) + a_{-1}B_2\left(\frac{3}{2}-3\right) + a_0B_2\left(\frac{3}{2}-4\right) + a_1B_2\left(\frac{3}{2}-5\right) = \\
a_{-1} \cdot 0 + a_0 \cdot 0 + a_1 \cdot \frac{1}{2} + a_2 \cdot \frac{1}{2} + a_{-1} \cdot 0 + a_0 \cdot 0 + a_1 \cdot 0 &= \frac{a_1 + a_2}{2}
\end{aligned}$$

$$\begin{aligned}
y_3 = s(2) &= a_{-1}B_2(2+1) + a_0B_2(2) + a_1B_2(2-1) + \\
&+ a_2B_2(2-2) + a_{-1}B_2(2-3) + a_0B_2(2-4) + a_1B_2(2-5) = \\
a_{-1} \cdot 0 + a_0 \cdot 0 + a_1 \cdot \left(\frac{1}{8}\right) + a_2 \cdot \frac{3}{4} + a_{-1} \cdot \left(\frac{1}{8}\right) + a_0 \cdot 0 + a_1 \cdot 0 &= \frac{1}{8}a_1 + \frac{3}{4}a_2 + \frac{1}{8}a_{-1}
\end{aligned}$$

The linear system in matrix form is:

$$\frac{1}{8} \cdot \begin{bmatrix} 4 & 4 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 6 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_{-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

The determinant of that matrix is 0 so the matrix is singular.

Point b)

The node coordinates to which fix the quadratic spline are $(0, y_0), (1, y_1), (2, y_2), (3, y_3)$.

Then, we can start formulating the equations for the linear system:

$$\begin{aligned}
y_0 = s(0) &= a_{-1}B_2(1) + a_0B_2(0) + a_1B_2(-1) + \\
&+ a_2B_2(-2) + a_{-1}B_2(-3) + a_0B_2(-4) + a_1B_2(-5) = \\
a_{-1} \cdot \frac{1}{8} + a_0 \cdot \frac{3}{4} + a_1 \cdot \frac{1}{8} + a_2 \cdot 0 + a_{-1} \cdot 0 + a_0 \cdot 0 + a_1 \cdot 0 &= \frac{1}{8}a_{-1} + \frac{3}{4}a_0 + \frac{1}{8}a_1
\end{aligned}$$

$$y_1 = s(1) = \frac{1}{8}a_0 + \frac{3}{4}a_1 + \frac{1}{8}a_2$$

$$y_2 = s(2) = \frac{1}{8}a_1 + \frac{3}{4}a_2 + \frac{1}{8}a_{-1}$$

$$\begin{aligned}
y_3 = s(3) &= a_{-1}B_2(3+1) + a_0B_2(3) + a_1B_2(3-1) + \\
&+ a_2B_2(3-2) + a_{-1}B_2(3-3) + a_0B_2(3-4) + a_1B_2(3-5) = \\
a_{-1} \cdot 0 + a_0 \cdot 0 + a_1 \cdot 0 + a_2 \cdot \frac{1}{8} + a_{-1} \cdot \frac{3}{4} + a_0 \cdot \frac{1}{8} + a_1 \cdot 0 &= \frac{1}{8}a_2 + \frac{3}{4}a_{-1} + \frac{1}{8}a_0
\end{aligned}$$

The linear system in matrix form is:

$$\frac{1}{8} \cdot \begin{bmatrix} 6 & 1 & 0 & 1 \\ 1 & 6 & 1 & 0 \\ 0 & 1 & 6 & 1 \\ 1 & 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_{-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Question 5

$$1 = y_0 = s(x_0) = 0 = a_{-1}B_3(1) + a_0B_3(0) + a_1B_3(-1) + a_2B_3(-2) + a_3B_3(-3) = \frac{1}{6}a_{-1} + \frac{2}{3}a_0 + \frac{1}{6}a_1$$

$$5 = y_1 = s(x_1) = 1 = a_{-1}B_3(2) + a_0B_3(1) + a_1B_3(0) + a_2B_3(-1) + a_3B_3(-2) = \frac{1}{6}a_0 + \frac{2}{3}a_1 + \frac{1}{6}a_2$$

$$1 = y_2 = s(x_2) = 2 = a_{-1}B_3(3) + a_0B_3(2) + a_1B_3(1) + a_2B_3(0) + a_3B_3(-1) = \frac{1}{6}a_1 + \frac{2}{3}a_2 + \frac{1}{6}a_3$$

$$s''(0) = a_{-1}B_3''(1) + a_0B_3''(0) + a_1B_3''(-1) + a_2B_3''(-2) + a_3B_3''(-3) = a_{-1} - 2a_0 + a_1$$

$$s''(2) = a_{-1}B_3''(3) + a_0B_3''(2) + a_1B_3''(1) + a_2B_3''(-0) + a_3B_3''(-1) = a_1 - 2a_2 + a_3$$

$$\frac{1}{6} \cdot \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} a_{-1} \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ y_0 \\ y_1 \\ y_2 \\ 0 \end{bmatrix}$$

We then use Gaussian *ellimination* to solve the system.

$$\begin{array}{cccc|cccc|cccc|cccc} 1 & -2 & 1 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 6 & 0 & 6 & 0 & 0 & 0 & 6 & 0 & 1 & 4 & 1 & 0 & 30 \\ 0 & 1 & 4 & 1 & 0 & 30 & 0 & 1 & 4 & 1 & 0 & 30 & 0 & 6 & 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 4 & 1 & 6 & 0 & 0 & 1 & 4 & 1 & 6 & 0 & 0 & 1 & 4 & 1 & 6 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \end{array}$$

$$\begin{array}{cccc|cccc|cccc|cccc} 1 & 0 & 9 & 2 & 0 & 60 & 1 & 0 & 9 & 2 & 0 & 60 & 1 & 0 & 0 & -34 & -9 & 6 \\ 0 & 1 & 4 & 1 & 0 & 30 & 0 & 1 & 4 & 1 & 0 & 30 & 0 & 1 & 0 & -15 & -4 & 6 \\ 0 & 0 & -24 & -6 & 0 & -174 & 0 & 0 & 1 & 4 & 1 & 6 & 0 & 0 & 1 & 4 & 1 & 6 \\ 0 & 0 & 1 & 4 & 1 & 6 & 0 & 0 & -24 & -6 & 0 & -174 & 0 & 0 & 0 & 90 & 24 & -30 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 6 & 0 & -6 \end{array}$$

$$\begin{array}{cccc|cccc|cccc|cccc} 1 & 0 & 0 & -34 & -9 & 6 & 1 & 0 & 0 & 0 & 1/15 & -16/3 & 1 & 0 & 0 & 0 & 0 & -5 \\ 0 & 1 & 0 & -15 & -4 & 6 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 4 & 1 & 6 & 0 & 0 & 1 & 0 & -1/15 & 22/3 & 0 & 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 4/15 & -1/3 & 0 & 0 & 0 & 1 & 4/15 & -1/3 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 6 & 0 & -6 & 0 & 0 & 0 & 0 & 8/5 & -8 & 0 & 0 & 0 & 0 & 1 & -5 \end{array}$$

Therefore, the coefficients are $a_{-1} = a_3 = -5$, $a_0 = a_2 = 1$, and $a_1 = 7$.