

# Howework 3 – Introduction to Computational Science

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## Question 1

$$i = 1 \quad l_1 = \begin{bmatrix} 1 \\ 1 \\ 5 \\ -3 \end{bmatrix} \quad u_1 = [2 \quad 1 \quad 1 \quad -2]$$

$$A_2 = \begin{bmatrix} 2 & 1 & 1 & -2 \\ 2 & 2 & -2 & -1 \\ 10 & 4 & 23 & -8 \\ -6 & -2 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 1 & -2 \\ 2 & 1 & 1 & -2 \\ 10 & 5 & 5 & -10 \\ -6 & -3 & -3 & 6 \end{bmatrix} = \begin{bmatrix} & & & \\ & 1 & -3 & 1 \\ & -1 & 18 & 2 \\ & 1 & 7 & 0 \end{bmatrix}$$

$$i = 2 \quad l_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad u_2 = [ \quad 1 \quad -3 \quad 1 ]$$

$$A_3 = \begin{bmatrix} & 1 & -3 & 1 \\ & -1 & 18 & 2 \\ & 1 & 7 & 0 \end{bmatrix} - \begin{bmatrix} & 1 & -3 & 1 \\ & -1 & 3 & -1 \\ & 1 & -3 & 1 \end{bmatrix} = \begin{bmatrix} & & & \\ & 15 & 3 & \\ & 10 & -1 & \end{bmatrix}$$

$$i = 3 \quad l_3 = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix} \quad u_3 = [ \quad 15 \quad 3 ]$$

$$A_4 = \begin{bmatrix} & & & \\ & 15 & 3 & \\ & 10 & -1 & \end{bmatrix} - \begin{bmatrix} & 15 & 3 & \\ & 10 & 2 & \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & -3 \end{bmatrix}$$

$$i = 4 \quad l_4 = \begin{bmatrix} \\ \\ 1 \end{bmatrix} \quad u_4 = [ \quad \quad -3 ]$$

$$L = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 5 & -1 & 1 & \\ -3 & 1 & 2/3 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 1 & 1 & -2 \\ & 1 & -3 & 1 \\ & & 15 & 3 \\ & & & -3 \end{bmatrix}$$

$$Ly = B \Rightarrow \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 5 & -1 & 1 & \\ -3 & 1 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 36 \\ 18 \end{bmatrix}$$

$$y_1 = -1$$

$$y_2 = -3 - (-1) \cdot 1 = -2$$

$$y_3 = 36 - 1 \cdot 2 - (-1) \cdot 5 = 39$$

$$y_4 = 18 - \frac{2}{3} \cdot 39 - (-2) - (-3) \cdot (-1) = -9$$

$$Ux = Y \Rightarrow \begin{bmatrix} 2 & 1 & 1 & -2 \\ & 1 & -3 & 1 \\ & & 15 & 3 \\ & & & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 39 \\ -9 \end{bmatrix}$$

$$x_4 = 3$$

$$x_3 = \frac{39 - 3 - 3}{15} = 2$$

$$x_2 = \frac{-2 - 3 - (-3) \cdot 2}{1} = 1$$

$$x_1 = \frac{-1 - (-2) \cdot 3 - 1 \cdot 2 - 1 \cdot 1}{2} = 1$$

$$x = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

## Question 2

$$i = 1 \quad k = 4 \quad [4 \ 2 \ 3 \ 1]$$

$$l_1 = \begin{bmatrix} 1/8 \\ 1/4 \\ 1/2 \\ 1 \end{bmatrix} \quad u_1 = [32 \ 24 \ 10 \ 11]$$

$$A_2 = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 8 & 8 & 5 & 2 \\ 16 & 12 & 10 & 5 \\ 32 & 24 & 20 & 11 \end{bmatrix} - \begin{bmatrix} 4 & 3 & 5/2 & 11/8 \\ 8 & 6 & 5 & 11/4 \\ 16 & 12 & 10 & 11/2 \\ 32 & 24 & 20 & 11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1/2 & -3/8 \\ 0 & 2 & 0 & -3/4 \\ 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$i = 2 \quad k = 2 \quad p = [4 \ 2 \ 3 \ 1]$$

$$l_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad u_2 = [0 \ 2 \ 0 \ -3/4]$$

$$A_3 = \begin{bmatrix} 0 & 0 & -1/2 & -3/8 \\ 0 & 2 & 0 & -3/4 \\ 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -3/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1/2 & -3/8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$i = 3 \quad k = 4 \quad p = [4 \ 2 \ 1 \ 3]$$

$$l_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad u_3 = [0 \ 0 \ -1/2 \ -3/8]$$

$$A_4 = \begin{bmatrix} 0 & 0 & -1/2 & -3/8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -1/2 & -3/8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$i = 4 \quad k = 4 \quad p = [4 \ 2 \ 1 \ 3]$$

$$l_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad u_4 = [0 \ 0 \ 0 \ -1/2]$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$L = P * \begin{bmatrix} 1/8 & 0 & 1 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/8 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 32 & 24 & 20 & 11 \\ 0 & 2 & 0 & -3/4 \\ 0 & 0 & -1/2 & -3/8 \\ 0 & 0 & 0 & -1/2 \end{bmatrix}$$

## Question 4

$$A_1 = \begin{bmatrix} 1 & 4 & 8 & 3 \\ 4 & 20 & 40 & 28 \\ 8 & 40 & 89 & 71 \\ 3 & 28 & 71 & 114 \end{bmatrix}$$

$$i = 1 \quad l_1 = \begin{bmatrix} 1 \\ 4 \\ 8 \\ 3 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 4 & 8 & 3 \\ 4 & 20 & 40 & 28 \\ 8 & 40 & 89 & 71 \\ 3 & 28 & 71 & 114 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 8 & 3 \\ 4 & 16 & 32 & 12 \\ 8 & 32 & 64 & 24 \\ 3 & 12 & 24 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 16 \\ 8 & 25 & 47 \\ 16 & 47 & 105 \end{bmatrix}$$

$$i = 2 \quad l_2 = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 4 & 8 & 16 \\ 8 & 25 & 47 \\ 16 & 47 & 105 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 16 \\ 8 & 16 & 32 \\ 16 & 32 & 64 \end{bmatrix} = \begin{bmatrix} & & \\ 9 & 15 & \\ 15 & 41 & \end{bmatrix}$$

$$i = 3 \quad l_3 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} & & \\ 9 & 15 & \\ 15 & 41 & \end{bmatrix} - \begin{bmatrix} & & \\ 9 & 15 & \\ 15 & 25 & \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & 16 \end{bmatrix}$$

$$i = 4 \quad l_4 = \begin{bmatrix} \\ \\ 4 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & & & \\ 4 & 2 & & \\ 8 & 4 & 3 & \\ 3 & 8 & 5 & 4 \end{bmatrix}$$

## Question 5

### Point a)

First of all, to show that  $A_{1,1}$  is symmetric, we say:

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} = A = A^T = \begin{bmatrix} A_{1,1}^T & A_{2,1} \\ A_{1,2} & A_{2,2}^T \end{bmatrix}$$

Therefore we can say that

$$A_{1,1} = A_{1,1}^T$$

and thus  $A_{1,1}$  is shown to be symmetric.

Then, since  $A$  is positive definite,  $> 0$  for any vector  $x \in R^n$ . We choose a vector  $x$  such that  $x_k = 0$  for any  $p < k \leq n$ .

Then:

$$x^T Ax = \sum_{j=1}^n x_j \cdot \sum_{i=1}^n x_i a_{i,j} = \sum_{j=1}^p x_j \cdot \sum_{i=1}^n x_i a_{i,j} + \sum_{j=p+1}^n 0 \cdot \sum_{i=1}^n 0 a_{i,j} = \sum_{j=1}^p x_j \cdot \left( \sum_{i=i}^p x_i a_{i,j} + \sum_{i=p+1}^n 0 a_{i,j} \right) =$$

$$= \sum_{j=1}^p x_j \cdot \sum_{i=i}^p x_i a_{i,j} = [x_1 \ x_2 \ \dots \ x_n] A_{1,1} [x_1 \ x_2 \ \dots \ x_n]^T$$

Then:

$$[x_1 \ x_2 \ \dots \ x_n] A_{1,1} [x_1 \ x_2 \ \dots \ x_n]^T > 0$$

which is the definition of positive definiteness for  $A_{1,1}$ .