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## Solution for Project 1

Due date: Thursday, 8 October 2020, 12:00 AM

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ewline

### Numerical Computing 2020 — Submission Instructions

(Please, notice that following instructions are mandatory:  
submissions that don't comply with, won't be considered)

- Assignments must be submitted to iCorsi (i.e. in electronic format).
- Provide both executable package and sources (e.g. C/C++ files, Matlab). If you are using libraries, please add them in the file. Sources must be organized in directories called:  
*Project\_number\_lastname\_firstname*  
and the file must be called:  
*project\_number\_lastname\_firstname.zip*  
*project\_number\_lastname\_firstname.pdf*
- The TAs will grade your project by reviewing your project write-up, and looking at the implementation you attempted, and benchmarking your code's performance.
- You are allowed to discuss all questions with anyone you like; however: (i) your submission must list anyone you discussed problems with and (ii) you must write up your submission independently.

The purpose of this assignment<sup>1</sup> is to learn the importance of numerical linear algebra algorithms to solve fundamental linear algebra problems that occur in search engines.

## 1. Page-Rank Algorithm

### 1.1. Theory [20 points]

#### 1.1.1. Show that the order of convergence of the power method is linear, and state what the asymptotic error constant is.

First of all, we show the the sequence of vectors computed by power iteration indeed converges to  $\lambda_1$  or the biggest eigenvector (we assume we name eigenvectors in decreasing order of magnitude, with  $|\lambda_1| > |\lambda_i|$  for  $i \in 2..n$ ).

We can express the seed for the eigenvector (i.e. the initial value of  $v$  of the power iteration) as a linear combination of eigenvalues:

$$v_0 = \sum_{i=1}^n a_i x_i$$

We can then express the result of the n-th power method as

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<sup>1</sup>This document is originally based on a SIAM book chapter from *Numerical Computing with Matlab* from Clever B. Moler.

$$v_n = \gamma A v_{n-1} = A^n v_0 = \sum_{i=1}^n \gamma a_i \lambda_i^n x_i = \lambda_1^n \sum_{i=1}^n \gamma a_i \left(\frac{\lambda_i}{\lambda_1}\right)^n x_i = \gamma a_1 \lambda_1^n x_1 + \lambda_1^n \sum_{i=2}^n \gamma a_i \left(\frac{\lambda_i}{\lambda_1}\right)^n x_i$$

Here,  $\gamma$  is just a normalization term to make  $\|v_n\| = 1$ .  $v_n$  clearly converges to  $x_1$  since all the terms in the  $\sum_{i=2}^n$  contain  $\frac{\lambda_i}{\lambda_1}$ , which is always less than 1 if  $i > 1$  for the sorting of eigenvalues we did before. Therefore, these terms to the power of  $n$  converge to 0, and  $\gamma$  will cancel out  $a_1 \lambda_1^n$  due to the normalization, thus making the sequence converge to  $\lambda_1$ .

To see if the sequence converges linearly we use the definitions of rate of convergence:

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - \lambda_1|}{|x_n - \lambda_1|^1} = \mu$$

If this limit has a finite solution then the sequence converges linearly with rate  $\mu$ .

$$\lim_{n \rightarrow \infty} \frac{\left| a_1 \lambda_1^{n+1} x_1 + \lambda_1^{n+1} \sum_{i=2}^n a_i \left(\frac{\lambda_i}{\lambda_1}\right)^{n+1} x_i - \beta_{n+1} x_1 \right|}{\left| a_1 \lambda_1^n x_1 + \lambda_1^n \sum_{i=2}^n a_i \left(\frac{\lambda_i}{\lambda_1}\right)^n x_i - \beta_n x_1 \right|^1} = \mu$$

To simplify calculations, we consider the sequence without the normalization factor  $\gamma$  that will converge to a denormalized version of  $x_1$ , named  $\beta x_1$ . We can then simplify the  $a_1 \lambda_1^n x_1$  terms in the sequences with  $\beta_i x_1$  since  $\beta_i$  can be set freely.

Now we consider that if  $|\lambda_2| > |\lambda_i| \forall i \in \{3..n\}$  (since we sorted the eigenvalues), then  $\left(\frac{\lambda_i}{\lambda_1}\right)^n$  for  $i > 2$  will always converge faster to 0 than  $\left(\frac{\lambda_2}{\lambda_1}\right)^n$  thus all terms other than  $i = 2$  can be ignored in the limit computation. Therefore, the limit has finite solution and the convergence rate is

$$\mu = \frac{\lambda_2}{\lambda_1}$$

### 1.1.2. What assumptions should be made to guarantee convergence of the power method?

The first assumption to make is that the biggest eigenvalue in terms of absolute values should (let's name it  $\lambda_1$ ) be strictly greater than all other eigenvalues, so:

$$|\lambda_1| < |\lambda_i| \forall i \in \{2..n\}$$

Also, the eigenvector *guess* from which the power iteration starts must have a component in the direction of  $x_1$ , the eigenvector for the eigenvalue  $\lambda_1$  from before.

### 1.1.3. What is a shift and invert approach?

The shift and invert approach is a variant of the power method that may significantly increase the rate of convergence where some application of the vanilla method require large numbers of iterations. This improvement is achieved by taking the input matrix  $A$  and deriving a matrix  $B$  defined as:

$$B = (A - \alpha I)^{-1}$$

where  $\alpha$  is an arbitrary constant that must be chosen wisely in order to increase the rate of convergence. Since the eigenvalues  $u_i$  of  $B$  can be derived from the eigenvalues  $\lambda_i$  of  $A$ , namely:

$$u_i = \frac{1}{\lambda_i - \alpha}$$

the rate of convergence of the power method on  $B$  is:

$$\left| \frac{u_2}{u_1} \right| = \left| \frac{\frac{1}{\lambda_2 - \alpha}}{\frac{1}{\lambda_1 - \alpha}} \right| = \left| \frac{\lambda_1 - \alpha}{\lambda_2 - \alpha} \right|$$

By choosing  $\alpha$  close to  $\lambda_1$ , the convergence is sped up. To further increase the rate of convergence (up to a cubic rate), a new  $\alpha$ , and thus a new  $B$ , may be chosen for every iteration.

#### 1.1.4. What is the difference in cost of a single iteration of the power method, compared to the inverse iteration?

Inverse iteration is generally more expensive than a regular application of the power method, due to the overhead caused by the intermediate matrix  $B$ . One must either recompute  $B$  every time  $\alpha$  changes, which is rather expensive due to the inverse operation in the definition of  $B$ , or one must solve the matrix equation  $(A - \alpha I)v_k = v_{k-1}$  in every iteration.

#### 1.1.5. What is a Rayleigh quotient and how can it be used for eigenvalue computations?

The Rayleigh quotient is an effective way to either compute the corresponding eigenvalue of an eigenvector or the corresponding eigenvalue approximation of an eigenvector approximation. I.e., if  $x$  is an eigenvector, then:

$$\lambda = \mu(x) = \frac{x^T A x}{x^T x}$$

is the corresponding eigenvalue, while if  $x$  is an eigenvector approximation, for example found through some iterations of the power method, then  $\lambda$  is the closest possible approximation to the corresponding eigenvalue in a least-square sense.

### 1.2. Other webgraphs [10 points]

The provided PageRank MATLAB implementation was run 3 times on the starting websites <http://atelier.inf.usi.ch/>, [maggicl, https://www.iisbadoni.edu.it](https://www.iisbadoni.edu.it), and <https://www.usi.ch>, with results listed respectively in Figure ??, Figure ?? and Figure ??.

One pattern that emerges on the first and third execution is the presence of 1s in the main diagonal. This indicates that several pages found have a link to themselves.

Another interesting pattern, this time observable in all executions, is the presence of contiguous rectangular regions filled with 1s, especially along the main diagonal. This may be due to the presence of pages belonging to the same website, thus having a common layout and perhaps linking to a common set of internal (when near to the main diagonal) or external pages.

Finally, we can always observe a line starting from the top-left of  $G$  and ending in its bottom-left, running along a steep path slightly going right. This may be a side effect of the way pages are discovered and numbered: if new pages are continuously discovered, these pages will be added at the end of  $U$  and a corresponding vertical strip on 1s will appear in the bottomest non-colored region of  $G$ . This continues until  $n$  unique pages are visited and the line reaches the bottom edge of the connectivity matrix. The steepness of the line thus formed depends on the amount of new pages discovered in each of the first iterations of the `surfer(...)` function.

### 1.3. Connectivity matrix and subcliques [10 points]

The following ETH organization are following for the near cliques along the diagonal of the connectivity matrix in `eth500.mat`. The clique approximate position on the diagonal is indicated through the ranges in parenthesis.

- `baug.ethz.ch` (74-100)
- `mat.ethz.ch` (114-129)
- `mavt.ethz.ch` (164-182)

- `biol.ethz.ch` (198-216)
- `chab.ethz.ch` (221-236)
- `math.ethz.ch` (264-278)
- `erdw.ethz.ch` (321-337)
- `usys.ethz.ch` (358-373)
- `mtec.ethz.ch` (396-416)
- `gess.ethz.ch` (436-462)

## 1.4. Connectivity matrix and disjoint subgraphs [10 points]

### 1.4.1. What is the connectivity matrix $G$ (w.r.t figure 5)?

The connectivity matrix  $G$ , with  $U$  being defined as  $\{ "alpha", "beta", "gamma", "delta", "rho", "sigma" \}$  is:

$$G = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

### 1.4.2. What are the PageRanks if the hyperlink transition probability $p$ is the default value 0.85?

First we compute the matrix  $A$ , finding:

$$A = \frac{1}{40} \begin{bmatrix} 1 & 1 & 1 & 35 & 1 & 1 \\ 18 & 1 & 1 & 1 & 1 & 1 \\ 18 & 18 & 1 & 1 & 1 & 1 \\ 1 & 18 & 35 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 35 \\ 1 & 1 & 1 & 1 & 35 & 1 \end{bmatrix}$$

We then find the eigenvectors and eigenvalues of  $A$  through MATLAB, finding that the solution of  $Ax = 1x$  is:

$$x' \approx \begin{bmatrix} 0.4771 \\ 0.2630 \\ 0.3747 \\ 0.4905 \\ 0.4013 \\ 0.4013 \end{bmatrix}$$

To obtain more easily interpretable PageRank values, we can require that the sum of all PageRanks should be one. Once we normalize in this way, the result is:

$$x = \frac{x'}{\sum_{i=1}^6 x_i} \approx \begin{bmatrix} 0.1981 \\ 0.1092 \\ 0.1556 \\ 0.2037 \\ 0.1667 \\ 0.1667 \end{bmatrix}$$

Thus the pageranks are the components of vector  $x$ , w.r.t. the order given in  $U$ .

**1.4.3. Describe what happens with this example to both the definition of PageRank and the computation done by pagerank in the limit  $p \rightarrow 1$ .**

If  $p$  is closer to 1, then the probability a web user will visit a certain page randomly decreases, thus giving more weight in the computation of PageRank to the links between one page and another.

In the computation, increasing  $p$  decreases  $\delta$  (which represents the probability of a user randomly visiting a page), eventually making it 0 when  $p$  is 1.

**1.5. PageRanks by solving a sparse linear system [50 points]**

**1.5.1. Create pagerank1(G) by modifying pagerank.m to use the power method instead of solving the sparse linear system. What is an appropriate test for terminating the power iteration?**

The MATLAB solution to this question can be found on `files_data/pagerank1.m`.

An appropriate termination condition is to stop iterating when the sequence of solution vectors stops converging. This can be achieved by checking when, for an iteration  $n$

$$\|x_n - x_{n-1}\| < \|x_{n-1} - x_{n-2}\|$$

stops holding.

Since this condition might be too aggressive for a reasonably approximated solution, a cutoff value for the norm (say  $10^{-8}$ ) can also be introduced.

**1.5.2. Create pagerank2(G) by modifying pagerank.m to use the inverse iteration. Set  $\alpha$  equal to 0.8, 0.9 and 1 and comment on the different number of iterations they take until convergence. Also, what should be done in the unlikely event that the backslash operation involves a division by zero?**

The MATLAB solution to this question can be found on `files_data/pagerank1.m`.

The termination condition for the previous exercise is used also in this implementation.

Iterations are minimal with  $\alpha = 1$ . For example, for the graph in Figure 5 in the assignment, we have 5 iterations for  $\alpha = 1$ , 8 iterations for  $\alpha = 0.9$ , and 11 iterations for  $\alpha = 0.8$ .

To avoid divisions by 0 or near-0 numbers we check the reciprocal condition number of the matrix  $A - \alpha * I$ . If this number is below `eps`,  $\alpha$  must be changed. My implementation simply increases  $\alpha$  in increments of  $10^{-2}$  until a condition number higher than `eps` is found.

**1.5.3. Use your functions pagerank1.m and pagerank2.m (set  $\alpha = 1$ ) to compute the PageRanks of the six-node example presented in Figure 1. Make sure you get the correct result from each of your three functions.**

Here are the results for `pagerank1.m`:

Using power method implementation

```
page-rank  in  out  url
1  0.3210  2   2  alpha
6  0.2007  2   1  sigma
2  0.1705  1   2  beta
4  0.1368  2   1  delta
3  0.1066  1   3  gamma
5  0.0643  1   0  rho
```

And here are the results for `pagerank2.m`:

Using inverse iteration implementation

```
page-rank  in  out  url
```

1	0.3210	2	2	alpha
6	0.2007	2	1	sigma
2	0.1705	1	2	beta
4	0.1368	2	1	delta
3	0.1066	1	3	gamma
5	0.0643	1	0	rho

**1.5.4. Use your functions pagerank1.m and pagerank2.m (set  $\alpha = 1$ ) to compute the PageRanks of the six-node example presented in Figure 5. Discuss the differences between the results obtained now and the ones obtained in question 4.**

Here are the results for pagerank1.m:

Using power method implementation

	page-rank	in	out	url
4	0.2037	2	1	delta
1	0.1981	1	2	alpha
5	0.1667	1	1	rho
6	0.1667	1	1	sigma
3	0.1556	2	1	gamma
2	0.1092	1	2	beta

And here are the results for pagerank2.m:

Using inverse iteration implementation

	page-rank	in	out	url
4	0.2037	2	1	delta
1	0.1981	1	2	alpha
5	0.1667	1	1	rho
6	0.1667	1	1	sigma
3	0.1556	2	1	gamma
2	0.1092	1	2	beta

Both results are almost identical, with differences in the order of  $10^{-8}$  due to the approximation introduced by the termination conditions. This results are also approximately equal to the ones obtained manually once the normalization process by the sum of components is applied.

**1.5.5. Use your functions pagerank1.m and pagerank2.m (set  $\alpha = 1$ ) to compute the PageRanks of the three selected graphs from exercise 2. Report on the convergence of e.g. the power iteration for these subgraphs and summarize the advantage of the power method implemented in pagerank2.m against the original implementation in pagerank.m.**

In the following outputs, it = <number> represents the number of iterations required to compute the PageRanks within the termination criteria described before.

Here are the results for pagerank1.m for starting website <http://atelier.inf.usi.ch/~maggic1>:

Using power method implementation

it = 73

	page-rank	in	out	url
360	0.0869	31	1	<a href="https://creativecommons.org/licenses/by-sa/3.0">https://creativecommons.org/licenses/by-sa/3.0</a>
204	0.0406	8	1	<a href="https://forum.gitlab.com">https://forum.gitlab.com</a>
82	0.0189	117	18	<a href="https://www.mediawiki.org">https://www.mediawiki.org</a>

81	0.0188	117	4	<a href="https://wikimediafoundation.org">https://wikimediafoundation.org</a>
87	0.0150	6	1	<a href="https://docs.gitea.io">https://docs.gitea.io</a>
78	0.0145	114	9	<a href="https://www.mediawiki.org/wiki/Special:MyLanguage/How_to_contribute">https://www.mediawiki.org/wiki/Special:MyLanguage/How_to_contribute</a>
77	0.0132	77	13	<a href="https://foundation.wikimedia.org/wiki/Privacy_policy">https://foundation.wikimedia.org/wiki/Privacy_policy</a>
217	0.0127	40	8	<a href="https://bugs.archlinux.org">https://bugs.archlinux.org</a>
80	0.0115	107	6	<a href="https://foundation.wikimedia.org/wiki/Cookie_statement">https://foundation.wikimedia.org/wiki/Cookie_statement</a>
215	0.0114	38	5	<a href="https://bbs.archlinux.org">https://bbs.archlinux.org</a>
216	0.0114	38	8	<a href="https://wiki.archlinux.org">https://wiki.archlinux.org</a>
218	0.0114	38	6	<a href="https://security.archlinux.org">https://security.archlinux.org</a>
428	0.0107	9	1	<a href="https://www.dnb.de/kataloghilfe">https://www.dnb.de/kataloghilfe</a>
219	0.0102	38	7	<a href="https://aur.archlinux.org">https://aur.archlinux.org</a>
359	0.0098	9	1	<a href="https://creativecommons.org/publicdomain/zero/1.0">https://creativecommons.org/publicdomain/zero/1.0</a>
366	0.0092	27	21	<a href="https://archive.org">https://archive.org</a>
446	0.0089	24	5	<a href="https://foundation.wikimedia.org/wiki/Terms_of_Use">https://foundation.wikimedia.org/wiki/Terms_of_Use</a>
83	0.0079	78	0	<a href="https://schema.org">https://schema.org</a>
85	0.0074	77	0	<a href="https://www.wikimedia.org/static/images/wmf-hor-googpub.png">https://www.wikimedia.org/static/images/wmf-hor-googpub.png</a>
181	0.0066	8	2	<a href="https://gitlab.com">https://gitlab.com</a>
95	0.0062	2	1	<a href="https://www.enable-javascript.com">https://www.enable-javascript.com</a>
113	0.0061	13	1	<a href="https://www.britannica.com/topic/polenta">https://www.britannica.com/topic/polenta</a>
429	0.0058	8	1	<a href="https://www.dnb.de/EN/Home/home_node.html">https://www.dnb.de/EN/Home/home_node.html</a>
432	0.0058	8	1	<a href="https://www.dnb.de/expertensuche">https://www.dnb.de/expertensuche</a>
417	0.0058	8	1	<a href="https://www.dnb.de/DE/Home/home_node.html">https://www.dnb.de/DE/Home/home_node.html</a>
379	0.0057	24	2	<a href="https://blog.archive.org">https://blog.archive.org</a>
213	0.0057	32	7	<a href="https://www.archlinux.org">https://www.archlinux.org</a>
99	0.0056	3	1	<a href="https://www.usi.ch/it">https://www.usi.ch/it</a>
19	0.0051	4	1	<a href="https://creativecommons.org/licenses/by-nc-sa/4.0">https://creativecommons.org/licenses/by-nc-sa/4.0</a>
214	0.0050	31	5	<a href="https://www.archlinux.org/packages">https://www.archlinux.org/packages</a>
220	0.0050	31	6	<a href="https://www.archlinux.org/download">https://www.archlinux.org/download</a>

Here are the results for pagerank2.m for starting website <http://atelier.inf.usi.ch/~maggicl:>

Using inverse iteration implementation

it = 7

	page-rank	in	out	url
360	0.0869	31	1	<a href="https://creativecommons.org/licenses/by-sa/3.0">https://creativecommons.org/licenses/by-sa/3.0</a>
204	0.0406	8	1	<a href="https://forum.gitlab.com">https://forum.gitlab.com</a>
82	0.0189	117	18	<a href="https://www.mediawiki.org">https://www.mediawiki.org</a>
81	0.0188	117	4	<a href="https://wikimediafoundation.org">https://wikimediafoundation.org</a>
87	0.0150	6	1	<a href="https://docs.gitea.io">https://docs.gitea.io</a>
78	0.0145	114	9	<a href="https://www.mediawiki.org/wiki/Special:MyLanguage/How_to_contribute">https://www.mediawiki.org/wiki/Special:MyLanguage/How_to_contribute</a>
77	0.0132	77	13	<a href="https://foundation.wikimedia.org/wiki/Privacy_policy">https://foundation.wikimedia.org/wiki/Privacy_policy</a>
217	0.0127	40	8	<a href="https://bugs.archlinux.org">https://bugs.archlinux.org</a>
80	0.0115	107	6	<a href="https://foundation.wikimedia.org/wiki/Cookie_statement">https://foundation.wikimedia.org/wiki/Cookie_statement</a>
215	0.0114	38	5	<a href="https://bbs.archlinux.org">https://bbs.archlinux.org</a>
216	0.0114	38	8	<a href="https://wiki.archlinux.org">https://wiki.archlinux.org</a>
218	0.0114	38	6	<a href="https://security.archlinux.org">https://security.archlinux.org</a>
428	0.0107	9	1	<a href="https://www.dnb.de/kataloghilfe">https://www.dnb.de/kataloghilfe</a>
219	0.0102	38	7	<a href="https://aur.archlinux.org">https://aur.archlinux.org</a>
359	0.0098	9	1	<a href="https://creativecommons.org/publicdomain/zero/1.0">https://creativecommons.org/publicdomain/zero/1.0</a>

366	0.0092	27	21	<a href="https://archive.org">https://archive.org</a>
446	0.0089	24	5	<a href="https://foundation.wikimedia.org/wiki/Terms_of_Use">https://foundation.wikimedia.org/wiki/Terms_of_Use</a>
83	0.0079	78	0	<a href="https://schema.org">https://schema.org</a>
85	0.0074	77	0	<a href="https://www.wikimedia.org/static/images/wmf-hor-googpub.png">https://www.wikimedia.org/static/images/wmf-hor-googpub.png</a>
181	0.0066	8	2	<a href="https://gitlab.com">https://gitlab.com</a>
95	0.0062	2	1	<a href="https://www.enable-javascript.com">https://www.enable-javascript.com</a>
113	0.0061	13	1	<a href="https://www.britannica.com/topic/polenta">https://www.britannica.com/topic/polenta</a>
429	0.0058	8	1	<a href="https://www.dnb.de/EN/Home/home_node.html">https://www.dnb.de/EN/Home/home_node.html</a>
432	0.0058	8	1	<a href="https://www.dnb.de/expertensuche">https://www.dnb.de/expertensuche</a>
417	0.0058	8	1	<a href="https://www.dnb.de/DE/Home/home_node.html">https://www.dnb.de/DE/Home/home_node.html</a>
379	0.0057	24	2	<a href="https://blog.archive.org">https://blog.archive.org</a>
213	0.0057	32	7	<a href="https://www.archlinux.org">https://www.archlinux.org</a>
99	0.0056	3	1	<a href="https://www.usi.ch/it">https://www.usi.ch/it</a>
19	0.0051	4	1	<a href="https://creativecommons.org/licenses/by-nc-sa/4.0">https://creativecommons.org/licenses/by-nc-sa/4.0</a>
220	0.0050	31	6	<a href="https://www.archlinux.org/download">https://www.archlinux.org/download</a>
214	0.0050	31	5	<a href="https://www.archlinux.org/packages">https://www.archlinux.org/packages</a>

Here are the results for pagerank1.m for starting website <https://www.iisbadoni.edu.it/>:

Using power method implementation

it = 75

	page-rank	in	out	url
411	0.0249	42	1	<a href="https://twitter.com/mozilla">https://twitter.com/mozilla</a>
63	0.0248	145	1	<a href="https://twitter.com/firefox">https://twitter.com/firefox</a>
68	0.0203	142	1	<a href="https://www.instagram.com/firefox">https://www.instagram.com/firefox</a>
412	0.0164	37	1	<a href="https://www.instagram.com/mozilla">https://www.instagram.com/mozilla</a>
62	0.0080	21	1	<a href="https://github.com/mozilla/kitsune">https://github.com/mozilla/kitsune</a>
81	0.0070	110	2	<a href="https://www.apple.com">https://www.apple.com</a>
384	0.0064	5	1	<a href="https://www.xfinity.com/privacy/policy/dns">https://www.xfinity.com/privacy/policy/dns</a>
4	0.0064	32	0	<a href="https://">https://</a>
377	0.0059	19	1	<a href="https://abouthome-snippets-service.readthedocs.io/en/latest/data_collection.html">https://abouthome-snippets-service.readthedocs.io/en/latest/data_collection.html</a>
393	0.0059	19	1	<a href="https://www.adjust.com/terms/privacy-policy">https://www.adjust.com/terms/privacy-policy</a>
410	0.0057	16	1	<a href="https://wiki.mozilla.org/Firefox/Data_Collection">https://wiki.mozilla.org/Firefox/Data_Collection</a>
400	0.0057	15	1	<a href="https://yandex.ru/legal/confidential">https://yandex.ru/legal/confidential</a>
396	0.0057	15	1	<a href="https://github.com/mozilla-mobile/firefox-ios/blob/master/Docs/MMA.md">https://github.com/mozilla-mobile/firefox-ios/blob/master/Docs/MMA.md</a>
5	0.0056	31	0	<a href="https://ssl">https://ssl</a>
3	0.0054	36	0	<a href="https://www.iisbadoni.edu.it/sites/default/files/favicon.ico">https://www.iisbadoni.edu.it/sites/default/files/favicon.ico</a>
6	0.0054	36	0	<a href="https://www.iisbadoni.edu.it/sites/default/files/logo.png">https://www.iisbadoni.edu.it/sites/default/files/logo.png</a>
208	0.0054	159	0	<a href="https://schema.org">https://schema.org</a>
74	0.0052	178	5	<a href="https://foundation.mozilla.org">https://foundation.mozilla.org</a>
72	0.0052	33	32	<a href="https://www.mozilla.org/privacy/websites/#cookies">https://www.mozilla.org/privacy/websites/#cookies</a>
23	0.0051	2	1	<a href="https://www.iisbadoni.edu.it/mad">https://www.iisbadoni.edu.it/mad</a>
300	0.0051	157	0	<a href="https://accounts.firefox.com">https://accounts.firefox.com</a>

Here are the results for pagerank2.m for starting website <https://www.iisbadoni.edu.it/>:

Using inverse iteration implementation



it = 7

	page-rank	in	out	url
411	0.0249	42	1	https://twitter.com/mozilla
63	0.0248	145	1	https://twitter.com/firefox
68	0.0203	142	1	https://www.instagram.com/firefox
412	0.0164	37	1	https://www.instagram.com/mozilla
62	0.0080	21	1	https://github.com/mozilla/kitsune
81	0.0070	110	2	https://www.apple.com
384	0.0064	5	1	https://www.xfinity.com/privacy/policy/dns
4	0.0064	32	0	https:
393	0.0059	19	1	https://www.adjust.com/terms/privacy-policy
377	0.0059	19	1	https://abouthome-snippets-service.readthedocs.io/en/latest/data_collection.html
410	0.0057	16	1	https://wiki.mozilla.org/Firefox/Data_Collection
400	0.0057	15	1	https://yandex.ru/legal/confidential
396	0.0057	15	1	https://github.com/mozilla-mobile/firefox-ios/blob/master/Docs/MMA.md
5	0.0056	31	0	https://ssl
3	0.0054	36	0	https://www.iisbadoni.edu.it/sites/default/files/favicon.ico
6	0.0054	36	0	https://www.iisbadoni.edu.it/sites/default/files/logo.png
208	0.0054	159	0	https://schema.org
74	0.0052	178	5	https://foundation.mozilla.org
72	0.0052	33	32	https://www.mozilla.org/privacy/websites/#cookies
23	0.0051	2	1	https://www.iisbadoni.edu.it/mad
300	0.0051	157	0	https://accounts.firefox.com

Here are the results for pagerank1.m for starting website <https://www.usi.ch>:

Using power method implementation

it = 66

	page-rank	in	out	url
55	0.0741	354	1	https://www.instagram.com/usiuniversity
53	0.0324	366	3	https://www.facebook.com/usiuniversity
299	0.0248	6	1	https://twitter.com/usi_en
329	0.0243	8	1	https://www.facebook.com/USIElab
308	0.0156	7	3	https://www.facebook.com/USIFinancialCommunication
60	0.0155	316	2	https://www.swissuniversities.ch
424	0.0144	96	1	https://it.bul.sbu.usi.ch
330	0.0123	6	4	https://www.facebook.com/USI.ITDxC
320	0.0122	7	1	https://www.facebook.com/usiimeg
56	0.0107	320	0	https://www.youtube.com/usiuniversity
5	0.0096	317	71	https://usi.ch
62	0.0090	319	18	https://search.usi.ch
337	0.0087	7	1	https://twitter.com/usissoftware
63	0.0080	303	19	https://desk.usi.ch
130	0.0077	25	0	https://www.swissuniversities.ch/it
54	0.0072	208	0	https://twitter.com/USI_university
323	0.0066	9	5	https://www.facebook.com/usiorientamento
150	0.0062	12	1	https://www.innosuisse.ch/inno/it/home.html

248	0.0061	10	1	<a href="https://www.facebook.com/usimdfc">https://www.facebook.com/usimdfc</a>
106	0.0060	132	8	<a href="https://newsletter.usi.ch/archive/en">https://newsletter.usi.ch/archive/en</a>
135	0.0057	201	0	<a href="https://schema.org">https://schema.org</a>
326	0.0057	6	1	<a href="https://www.facebook.com/usialloggimendrisio">https://www.facebook.com/usialloggimendrisio</a>
322	0.0055	6	1	<a href="https://www.facebook.com/USImem">https://www.facebook.com/USImem</a>
366	0.0054	6	1	<a href="https://www.instagram.com/usi_ics_lugano">https://www.instagram.com/usi_ics_lugano</a>
212	0.0054	12	3	<a href="https://www.facebook.com/usimt">https://www.facebook.com/usimt</a>
7	0.0051	211	32	<a href="https://search.usi.ch/it">https://search.usi.ch/it</a>
6	0.0051	204	0	<a href="https://www.usi.ch/sites/all/themes/usiclean/img/bollino-usi.svg">https://www.usi.ch/sites/all/themes/usiclean /img/bollino-usi.svg</a>
14	0.0051	204	62	<a href="https://www.usi.ch/originalnode/342">https://www.usi.ch/originalnode/342</a>
15	0.0051	204	57	<a href="https://www.usi.ch/originalnode/358">https://www.usi.ch/originalnode/358</a>
16	0.0051	204	62	<a href="https://www.usi.ch/originalnode/343">https://www.usi.ch/originalnode/343</a>
17	0.0051	204	57	<a href="https://www.usi.ch/originalnode/344">https://www.usi.ch/originalnode/344</a>
18	0.0051	204	58	<a href="https://www.usi.ch/en/originalnode/12174">https://www.usi.ch/en/originalnode/12174</a>
20	0.0051	204	60	<a href="https://www.usi.ch/originalnode/349">https://www.usi.ch/originalnode/349</a>
21	0.0051	204	62	<a href="https://www.usi.ch/originalnode/8996">https://www.usi.ch/originalnode/8996</a>
22	0.0051	204	60	<a href="https://www.usi.ch/originalnode/348">https://www.usi.ch/originalnode/348</a>
23	0.0051	204	59	<a href="https://www.usi.ch/originalnode/351">https://www.usi.ch/originalnode/351</a>
24	0.0051	204	58	<a href="https://www.usi.ch/originalnode/350">https://www.usi.ch/originalnode/350</a>
25	0.0051	204	61	<a href="https://www.usi.ch/originalnode/353">https://www.usi.ch/originalnode/353</a>
26	0.0051	204	58	<a href="https://www.usi.ch/en/originalnode/354">https://www.usi.ch/en/originalnode/354</a>
27	0.0051	204	59	<a href="https://www.usi.ch/originalnode/8014">https://www.usi.ch/originalnode/8014</a>
61	0.0051	204	0	<a href="https://www.usi.ch/sites/all/themes/usiclean/img/swissuniversities.svg">https://www.usi.ch/sites/all/themes/usiclean /img/swissuniversities.svg</a>
57	0.0050	188	9	<a href="https://newsletter.usi.ch/archive">https://newsletter.usi.ch/archive</a>

Here are the results for pagerank2.m for starting website <https://www.usi.ch>:

Using inverse iteration implementation

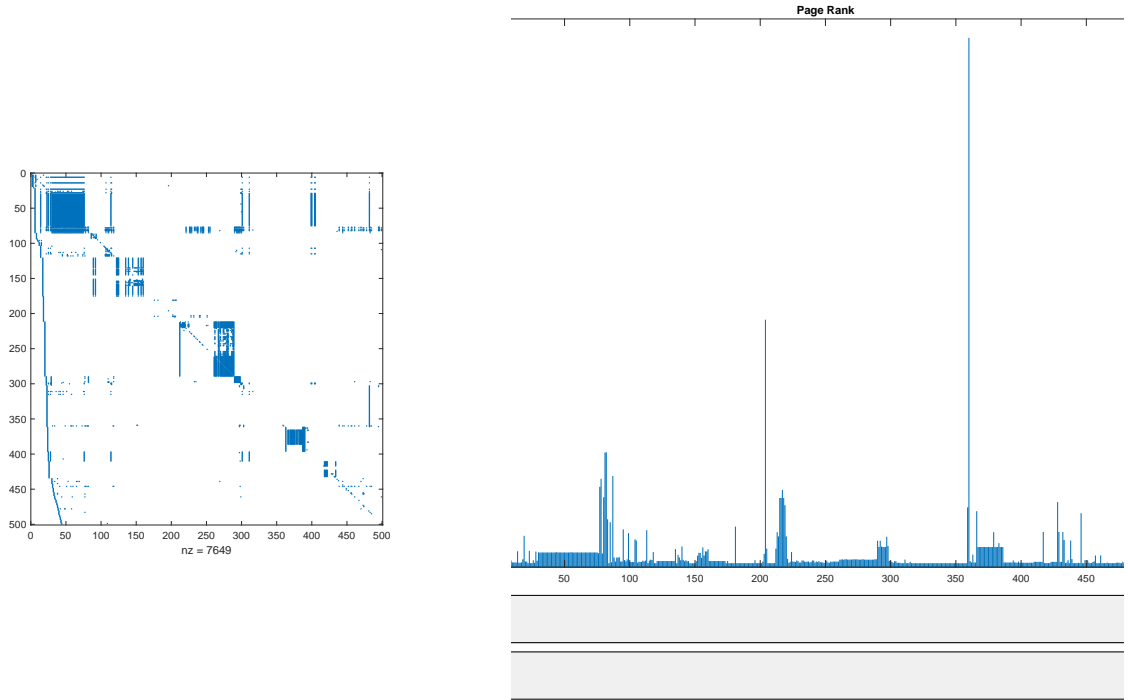
it = 7

	page-rank	in	out	url
55	0.0741	354	1	<a href="https://www.instagram.com/usiuniversity">https://www.instagram.com/usiuniversity</a>
53	0.0324	366	3	<a href="https://www.facebook.com/usiuniversity">https://www.facebook.com/usiuniversity</a>
299	0.0248	6	1	<a href="https://twitter.com/usi_en">https://twitter.com/usi_en</a>
329	0.0243	8	1	<a href="https://www.facebook.com/USIElab">https://www.facebook.com/USIElab</a>
308	0.0156	7	3	<a href="https://www.facebook.com/USIFinancialCommunication">https://www.facebook.com/USIFinancialCommunication</a>
60	0.0155	316	2	<a href="https://www.swissuniversities.ch">https://www.swissuniversities.ch</a>
424	0.0144	96	1	<a href="https://it.bul.sbu.usi.ch">https://it.bul.sbu.usi.ch</a>
330	0.0123	6	4	<a href="https://www.facebook.com/USI.ITDxC">https://www.facebook.com/USI.ITDxC</a>
320	0.0122	7	1	<a href="https://www.facebook.com/usiimeg">https://www.facebook.com/usiimeg</a>
56	0.0107	320	0	<a href="https://www.youtube.com/usiuniversity">https://www.youtube.com/usiuniversity</a>
5	0.0096	317	71	<a href="https://usi.ch">https://usi.ch</a>
62	0.0090	319	18	<a href="https://search.usi.ch">https://search.usi.ch</a>
337	0.0087	7	1	<a href="https://twitter.com/usisoftware">https://twitter.com/usisoftware</a>
63	0.0080	303	19	<a href="https://desk.usi.ch">https://desk.usi.ch</a>
130	0.0077	25	0	<a href="https://www.swissuniversities.ch/it">https://www.swissuniversities.ch/it</a>
54	0.0072	208	0	<a href="https://twitter.com/USI_university">https://twitter.com/USI_university</a>
323	0.0066	9	5	<a href="https://www.facebook.com/usiorientamento">https://www.facebook.com/usiorientamento</a>
150	0.0062	12	1	<a href="https://www.innosuisse.ch/inno/it/home.html">https://www.innosuisse.ch/inno/it/home.html</a>
248	0.0061	10	1	<a href="https://www.facebook.com/usimdfc">https://www.facebook.com/usimdfc</a>
106	0.0060	132	8	<a href="https://newsletter.usi.ch/archive/en">https://newsletter.usi.ch/archive/en</a>

135	0.0057	201	0	<a href="https://schema.org">https://schema.org</a>
326	0.0057	6	1	<a href="https://www.facebook.com/usialloggimendrisio">https://www.facebook.com/usialloggimendrisio</a>
322	0.0055	6	1	<a href="https://www.facebook.com/USImem">https://www.facebook.com/USImem</a>
366	0.0054	6	1	<a href="https://www.instagram.com/usi_ics_lugano">https://www.instagram.com/usi_ics_lugano</a>
212	0.0054	12	3	<a href="https://www.facebook.com/usimt">https://www.facebook.com/usimt</a>
7	0.0051	211	32	<a href="https://search.usi.ch/it">https://search.usi.ch/it</a>
14	0.0051	204	62	<a href="https://www.usi.ch/originalnode/342">https://www.usi.ch/originalnode/342</a>
27	0.0051	204	59	<a href="https://www.usi.ch/originalnode/8014">https://www.usi.ch/originalnode/8014</a>
21	0.0051	204	62	<a href="https://www.usi.ch/originalnode/8996">https://www.usi.ch/originalnode/8996</a>
22	0.0051	204	60	<a href="https://www.usi.ch/originalnode/348">https://www.usi.ch/originalnode/348</a>
25	0.0051	204	61	<a href="https://www.usi.ch/originalnode/353">https://www.usi.ch/originalnode/353</a>
61	0.0051	204	0	<a href="https://www.usi.ch/sites/all/themes/usiclean/img/swissuniversities.svg">https://www.usi.ch/sites/all/themes/usiclean /img/swissuniversities.svg</a>
16	0.0051	204	62	<a href="https://www.usi.ch/originalnode/343">https://www.usi.ch/originalnode/343</a>
23	0.0051	204	59	<a href="https://www.usi.ch/originalnode/351">https://www.usi.ch/originalnode/351</a>
15	0.0051	204	57	<a href="https://www.usi.ch/originalnode/358">https://www.usi.ch/originalnode/358</a>
17	0.0051	204	57	<a href="https://www.usi.ch/originalnode/344">https://www.usi.ch/originalnode/344</a>
24	0.0051	204	58	<a href="https://www.usi.ch/originalnode/350">https://www.usi.ch/originalnode/350</a>
6	0.0051	204	0	<a href="https://www.usi.ch/sites/all/themes/usiclean/img/bollino-usi.svg">https://www.usi.ch/sites/all/themes/usiclean /img/bollino-usi.svg</a>
20	0.0051	204	60	<a href="https://www.usi.ch/originalnode/349">https://www.usi.ch/originalnode/349</a>
26	0.0051	204	58	<a href="https://www.usi.ch/en/originalnode/354">https://www.usi.ch/en/originalnode/354</a>
18	0.0051	204	58	<a href="https://www.usi.ch/en/originalnode/12174">https://www.usi.ch/en/originalnode/12174</a>
57	0.0050	188	9	<a href="https://newsletter.usi.ch/archive">https://newsletter.usi.ch/archive</a>

The potential of the algorithm in `pagerank2.m` compared to the one in `pagerank.m` is that by using inverse iteration, thus theoretically avoiding a full matrix system solution process, the sparsity of matrix  $G$  can be maintained during the computation thus saving memory costs.

However, since my `pagerank2.m` implementation is a simplified implementation of the algorithm and indeed uses system solution (MATLAB's `mldivide`), this theoretical advantages are voided: my implementation is 3 times as long as the original one and uses approximately the same amount of memory. Advantages over `pagerank1.m` (due to the intentionally increased rate of convergence) are clear: in all the examples above, the inverse iteration implementation is at least 8 times faster than the “vanilla” power method.



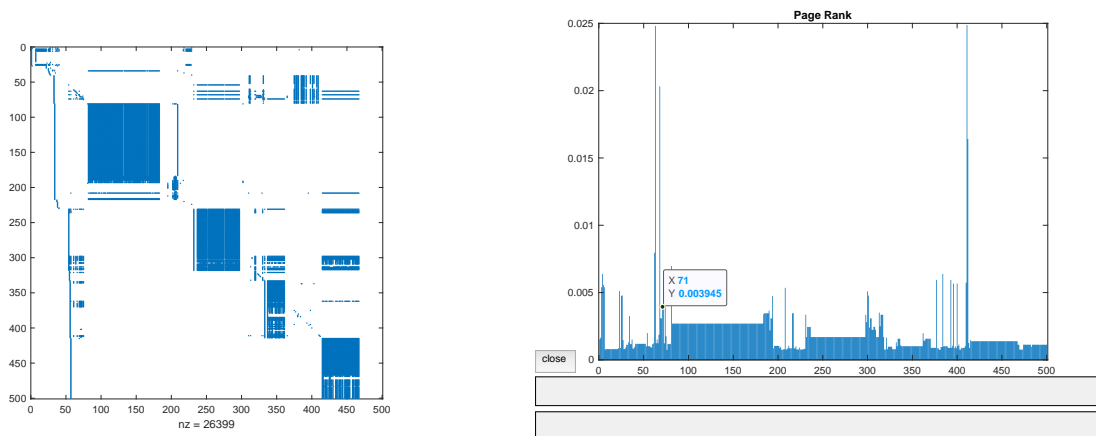
(a) Spy plot of connectivity matrix

(b) Page rank bar graph

360	0.0869	1	<a href="https://creativecommons.org/licenses/by-sa/3.0">https://creativecommons.org/licenses/by-sa/3.0</a>
204	0.0406	8	1 <a href="https://forum.gitlab.com">https://forum.gitlab.com</a>
82	0.01897		<a href="https://www.mediawiki.org">https://www.mediawiki.org</a>
81	0.01887	4	<a href="https://wikimediafoundation.org">https://wikimediafoundation.org</a>
87	0.0150	6	1 <a href="https://docs.gitea.io">https://docs.gitea.io</a>
78	0.01454	9	<a href="https://www.mediawiki.org/wiki/Special:MyLanguage/How_to_contribute">https://www.mediawiki.org/wiki/Special:MyLanguage/How_to_contribute</a>
77	0.0132		<a href="https://foundation.wikimedia.org/wiki/Privacy_policy">https://foundation.wikimedia.org/wiki/Privacy_policy</a>
217	0.0127	8	<a href="https://bugs.archlinux.org">https://bugs.archlinux.org</a>
80	0.01157	6	<a href="https://foundation.wikimedia.org/wiki/Cookie_statement">https://foundation.wikimedia.org/wiki/Cookie_statement</a>
215	0.0114	5	<a href="https://bbs.archlinux.org">https://bbs.archlinux.org</a>

(c) Top 10 webpages with highest PageRank

Figure 1: Results of first PageRank calculation (for starting website <http://atelier.inf.usi.ch/maggicl/>)



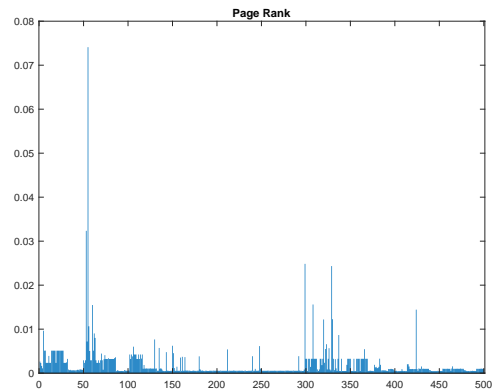
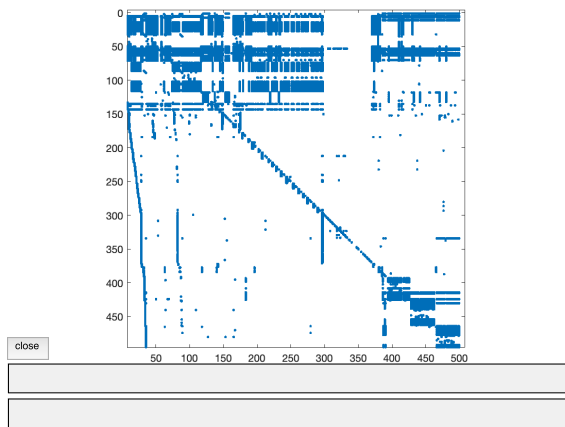
(a) Spy plot of connectivity matrix

(b) Page rank bar graph

411	0.0249	1	<a href="https://twitter.com/mozilla">https://twitter.com/mozilla</a>
63	0.02485	1	<a href="https://twitter.com/firefox">https://twitter.com/firefox</a>
68	0.02032	1	<a href="https://www.instagram.com/firefox">https://www.instagram.com/firefox</a>
412	0.0164	1	<a href="https://www.instagram.com/mozilla">https://www.instagram.com/mozilla</a>
62	0.0080	1	<a href="https://github.com/mozilla/kitsune">https://github.com/mozilla/kitsune</a>
81	0.00700	2	<a href="https://www.apple.com">https://www.apple.com</a>
384	0.0064	5	1 <a href="https://www.xfinity.com/privacy/policy/dns">https://www.xfinity.com/privacy/policy/dns</a>
4	0.0064	0	<a href="https://">https://</a>
377	0.0059	1	<a href="https://abouthome-snippets-service.readthedocs.io/en/latest/data_collection.html">https://abouthome-snippets-service.readthedocs.io/en/latest/data_collection.html</a>
393	0.0059	1	<a href="https://www.adjust.com/terms/privacy-policy">https://www.adjust.com/terms/privacy-policy</a>
410	0.0057	1	<a href="https://wiki.mozilla.org/Firefox/Data_Collection">https://wiki.mozilla.org/Firefox/Data_Collection</a>

(c) Top 10 webpages with highest PageRank

Figure 2: Results of second PageRank calculation (for starting website <https://www.iisbadoni.edu.it/>)



(a) Spy plot of connectivity matrix

(b) Page rank bar graph

55	0.07414	1	<a href="https://www.instagram.com/usiuniversity">https://www.instagram.com/usiuniversity</a>
53	0.03246	3	<a href="https://www.facebook.com/usiuniversity">https://www.facebook.com/usiuniversity</a>
299	0.0248	6	1 <a href="https://twitter.com/usi_en">https://twitter.com/usi_en</a>
329	0.0243	8	1 <a href="https://www.facebook.com/USIElab">https://www.facebook.com/USIElab</a>
308	0.0156	7	3 <a href="https://www.facebook.com/USIFinancialCommunication">https://www.facebook.com/USIFinancialCommunication</a>
60	0.01556	2	<a href="https://www.swissuniversities.ch">https://www.swissuniversities.ch</a>
424	0.0144	1	<a href="https://it.bul.sbu.usi.ch">https://it.bul.sbu.usi.ch</a>
330	0.0123	6	4 <a href="https://www.facebook.com/USI.ITDxC">https://www.facebook.com/USI.ITDxC</a>
320	0.0122	7	1 <a href="https://www.facebook.com/usiimeg">https://www.facebook.com/usiimeg</a>
56	0.01070	0	<a href="https://www.youtube.com/usiuniversity">https://www.youtube.com/usiuniversity</a>

(c) Top 10 webpages with highest PageRank

Figure 3: Results of third PageRank calculation (for starting website <https://www.usi.ch/>)