Università
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Institute of Computational Science ICS

Numerical Computing **2020**

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Solution for Project 6

Due date: Friday, December 18, 2020, 11:59 PM

The purpose of this mini-project is to implement the Simplex Method to find the solution of linear programs, involving both the minimisation and the maximisation of the objective function.

1. Graphical Solution of Linear Programming Problems [25 points]

Please consider the following two problems:

(1)

$$\begin{aligned} \min \quad & z = 4x + y \\ \text{s.t.} \quad & x + 2y \leq 40 \\ & x + y \geq 30 \\ & 2x + 3y \geq 72 \\ & x,\, y \geq 0 \end{aligned}$$

(2) A tailor plans to sell two types of trousers, with production costs of 25 CHF and 40 CHF, respectively. The former type can be sold for 85 CHF, while the latter for 110 CHF. The tailor estimates a total monthly demand of 265 trousers. Find the number of units of each type of trousers that should be produced in order to maximise the net profit of the tailor, if we assume that the he cannot spend more than 7000 CHF in raw materials.

Start by writing problem (2) as a linear programming problem. Then complete the following tasks:

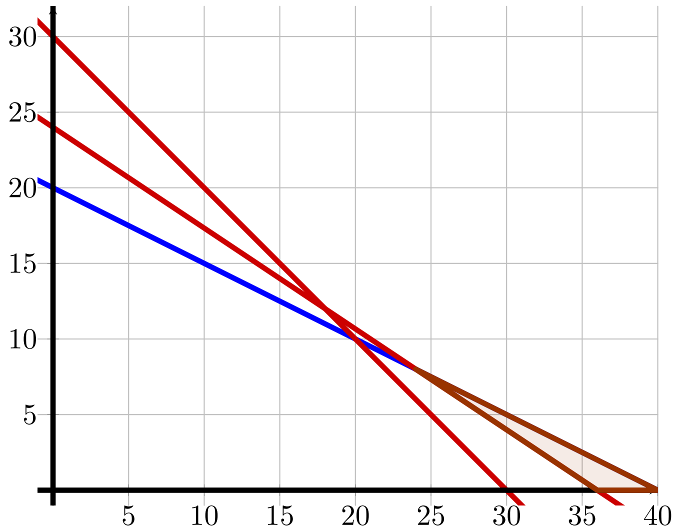
- Solve the system of inequalities.
- Plot the feasible region identified by the constraints.
- Find the optimal solution and the value of the objective function in that point.

1.1. Solving problem 1

We first solve the system of inequalities for y:

$$\begin{cases} y \leq 20 - \frac{1}{2}x \\ y \geq 30 - x \\ y \geq 24 - \frac{2}{3}x \\ x, y \geq 0 \end{cases}$$

Here is the feasability region plot, which was derived geometrically by plotting the inequalities above.



Red lines represent feasibility constraints keeping the upper part of the graph, while blue lines keep the low part. The resulting region is the area in brown.

We therefore identify the verticies $(24, 8)$, $(36, 0)$ and $(40, 0)$ as candidate solutions, and therefore we evaluate the minimization function in these points:

$$m(x, y) = 4x + y$$
$$m(24, 8) = 104$$
$$m(36, 0) = 144$$
$$m(40, 0) = 160$$

And therefore we have as a solution $(x, y) = (24, 8)$, with $z = 104$, as this is a minimization problem.

1.2. Problem 2 as linear programming problem

The following is the linear programming formulation of problem 2:

$$\max\ (85 - 25)x + (110 - 40)y$$

$$\text{s.t.} \begin{cases} 25x + 40y \leq 7000 \\ x + y \leq 265 \\ x, y \geq 0 \end{cases}$$

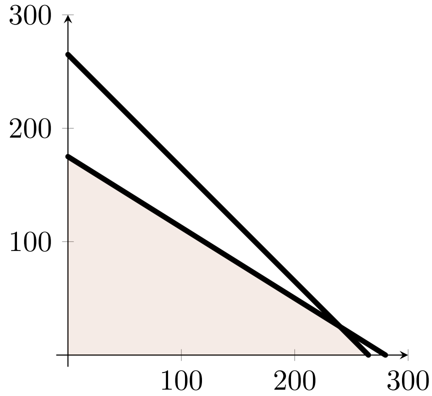
1.3. Solving problem 2

We first solve the system of inequalities for y:

$$\begin{cases} 25x + 40y \leq 7000 \\ x + y \leq 265 \\ x, y \geq 0 \end{cases} = \begin{cases} y < 175 - \frac{5}{8}x \\ x \leq 265 - x \\ x, y \geq 0 \end{cases}$$

From this solution it is clear that the region of satisfactory xs and ys is bounded by $x = 0$, $y = 0$, $y = 265 - x$, and $y = 175 - \frac{5}{8}x$.

Here is a plot of the feasability region:



We then proceed to evaluate the maximization function at all verticies of the feasability region, i.e. $\{(0,0), (265,0), (0,175), (240,25)\}$.

$$\begin{aligned} m(x, y) &= 60x + 70y \\ m(0,0) &= 0 \\ m(0,175) &= 12250 \\ m(240,25) &= 16150 \\ m(265,0) &= 15900 \end{aligned}$$

We then conclude the tailor should produce 240 trousers of the first type and 25 trousers of the second type, for a revenue of 16,150.– Fr.

2. Implementation of the Simplex Method [35 points]

The implementation of the simplex method can be found in the `Project_6_Claudio_Maggioni/` folder.

3. Applications to Real-Life Example: Cargo Aircraft [25 points]

In this second part of the assignment, you are required to use the simplex method implementation to solve a real-life problem taken from economics (constrained profit maximisation).

A cargo aircraft has 4 compartments (indicated simply as $S_1, \ldots, S_4$) used to store the goods to be transported. Details about the weight capacity and storage capacity of the different compartments can be inferred from the data reported in the following table:

Compartment	Weight Capacity (t)	Storage Capacity (m^3)
S_1	18	11930
S_2	32	22552
S_3	25	11209
S_4	17	5870

The following four cargos are available for shipment during the next flight:

Cargo	Weight (t)	Volume (m^3/t)	Profit (CHF/t)
C_1	16	320	135
C_2	32	510	200
C_3	40	630	410
C_4	28	125	520

Any proportion of the four cargos can be accepted, and the profit obtained for each cargo is increased by 10% if it is put in S_2, by 20% if it is put in S_3 and by 30% if it is put in S_4, due to the better storage conditions . The objective of this problem is to determine which amount of the different cargos will be transported and how to allocate it among the different compartments, while maximising the profit of the owner of the cargo plane.

3.1. Formulate the problem above as a linear program: what is the objective function? What are the constraints? Write down all equations, with comments explaining what you are doing.

From the problem description we can determine that this problem has 16 variables $x_{i,j}$, where $1 \leq i, j \leq 4$ $i, j \in N$. Each variable represents a weight in tonnes, with i determining the type of cargo used and j the compartment used. Therefore the variables of this problem to optimize represent how much cargo to put where and in which quantity.

From this initial discussion we can easily derive the objective function to maximize (as the objective function will output revenue, and we want to be greedy as most Swiss people):

$$\max z = \text{grandsum}\left[\left(\begin{bmatrix}135\\200\\410\\520\end{bmatrix} \cdot \begin{bmatrix}1 & 1.1 & 1.2 & 1.3\end{bmatrix}\right) \odot \begin{bmatrix}x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4}\\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4}\\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4}\\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4}\end{bmatrix}\right]$$

where $\odot$ is an element-wise or Hadamard product and grandsum(A) simply sums all elements of A. The column vector represents the value per ton of each good while the row vector represents the revenue multipliers associated with each cargo hold.

We can move to cargo hold constraints on weight, which check if the cargo holds are overloaded:

$$\sum_{i=1}^{4} x_{i,1} \leq 18 \qquad \sum_{i=1}^{4} x_{i,2} \leq 32 \qquad \sum_{i=1}^{4} x_{i,3} \leq 25 \qquad \sum_{i=1}^{4} x_{i,4} \leq 17$$

Then we consider compartment constraints on space, which check if the cargos will fit:

$$\begin{aligned}
320x_{1,1} + 510x_{2,1} + 630x_{3,1} + 125x_{4,1} &\leq 11930\\
320x_{1,2} + 510x_{2,2} + 630x_{3,2} + 125x_{4,2} &\leq 22552\\
320x_{1,3} + 510x_{2,3} + 630x_{3,3} + 125x_{4,3} &\leq 11209\\
320x_{1,4} + 510x_{2,4} + 630x_{3,4} + 125x_{4,4} &\leq 5870
\end{aligned}$$

Then we have a final set of constraints to check if we are loading more cargo than we own:

$$\sum_{i=1}^{4} x_{1,i} \leq 16 \qquad \sum_{i=1}^{4} x_{1,i} \leq 32 \qquad \sum_{i=1}^{4} x_{1,i} \leq 40 \qquad \sum_{i=1}^{4} x_{1,i} \leq 28$$

Therefore, we have a final minimization problem in standard form with 16 variables and 12 feasability constraints.

3.2. Create a script *exercise3.m* which uses the simplex method implemented in the previous exercise to solve the problem. What is the optimal solution? Visualise it graphically and briefly comment the results obtained (are you surprised of this outcome on the basis of your data?)

The file that solves this problem using our MATLAB's implementation of the simplex method can be found in the *exercise3.m* file.

The optimal solution according to the simplex method is the following:

$$\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 18 & 6 & 0 & 0 \\ 0 & 26 & 14 & 0 \\ 0 & 0 & 11 & 17 \end{bmatrix}$$

The revenue for this load is of 41890.- Fr.

Figure 1 shows a graphical representation of this solution. The solution seems quite intuitive: the simplex method seems to have found quite a greedy solution, placing the most valuable cargo in the "best" cargo hold, proceeding with less valuable cargos and "worse" cargo holds once either the filling constraints of the hold are reached or we run out of our most valuable cargo left.

It is notable that weight limits are reached before space limits, but in hindsight it is a quite likely scenario for a cargo plane. This problem has quite realistic data, as the total weight limit (82 tonnes) matches almost exactly the designed load capacity of a cargo variant Boeing 747 (81.6 tonnes[1]).

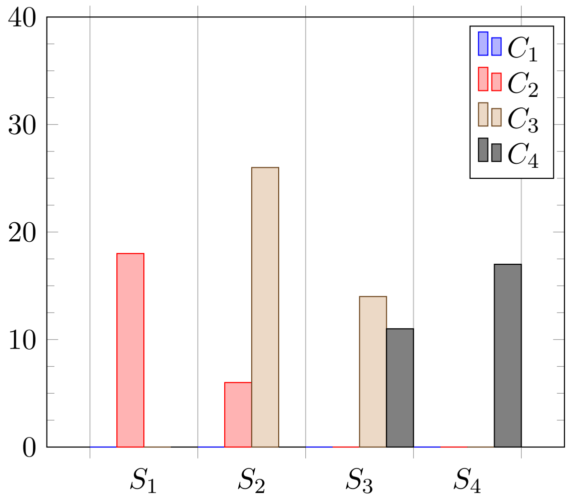


Figure 1: Solution to the Cargo Airplane as an histogram of tonnes loaded per cargo hold per cargo type. Y-axis is the amount tonnes loaded, X-axis is the cargo hold where the cargo should be loaded, bar color is the cargo type loaded.

4. Cycling and Degeneracy [15 points]

Consider now the following simple problem:

$$\begin{aligned} \max \quad & z = 3x_1 + 4x_2, \\ \text{s.t.} \quad & 4x_1 + 3x_2 \leq 12 \\ & 4x_1 + x_2 \leq 8 \\ & 4x_1 + 2x_2 \leq 8 \\ & x_1, x_2 \geq 0. \end{aligned}$$

[1]Source: Wikipedia at `https://en.wikipedia.org/wiki/Boeing_747#Background`

4.1. Create a script *exercise4.m* which uses the simplex method implemented above to solve this problem. Do you achieve convergence within the maximum number of iterations (given by the maximum number of possible basic solutions)? Do you notice any strange behaviour? (*hint:* check, e.g., the indices of the entering and departing variables)

The MATLAB implementation for this problem can be found under file *exercise4.m* in the resources folder (i.e. `Project_6_Claudio_Maggioni/`).

The simplex method implementation for this problem fails with an error in the auxiliary problem initialization. The error message is "The original LP problem does not admit a feasible solution". This error message appears when the simplex method does not converge in the expected maximum number of iterations. The large number of iterations (793 before forced termination) is caused by an "oscillation" or vicious cycle of swap operations between matrix B and matrix D. This oscillation is cause by the fact that entering and departing variable indexes do not change, and stay fixed at values 6 and 4 from the second iteration onwards. This causes an unnecessary continuous oscillation between two solutions that are equally not optimal ($z = 28$) according to the auxiliary problem definition, terminating the auxiliary algorithm abnormally after too many iterations.

4.2. Look at the number of constraints and at the number of unknowns: what can you notice about the underlying system of equations? Represent them graphically and try to use this information to explain the behaviour of your solver in the previous point.

In figure 2 lies a graphical representation of this linear programming problem.

The anomaly in the feasability region of this problem is that its perimeter has triangular shape, and it is made up only of one constraint other than the trivial positivity constraints at the axes (i.e. the constraints different from $4x + 2y \leq 8$ do not make a difference in the feasability region).

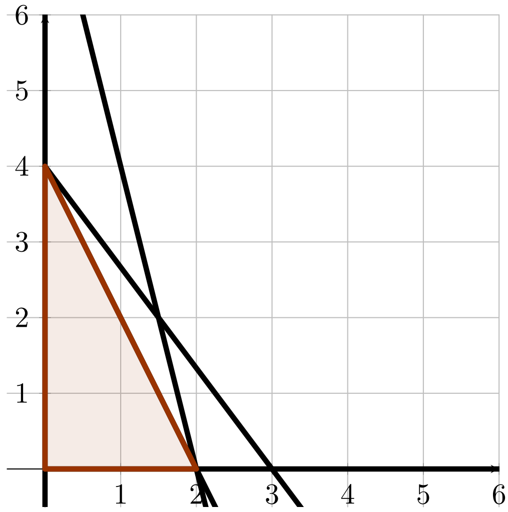


Figure 2: Feasability region plot of maximization problem for exercise 4. Note the fact that the region perimeter is a triangle.