Homework 5 – Optimization Methods

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Exercise 2

Exercise 2.1

The resulting MATLAB plot of each constraint and of the feasible region is shown below:
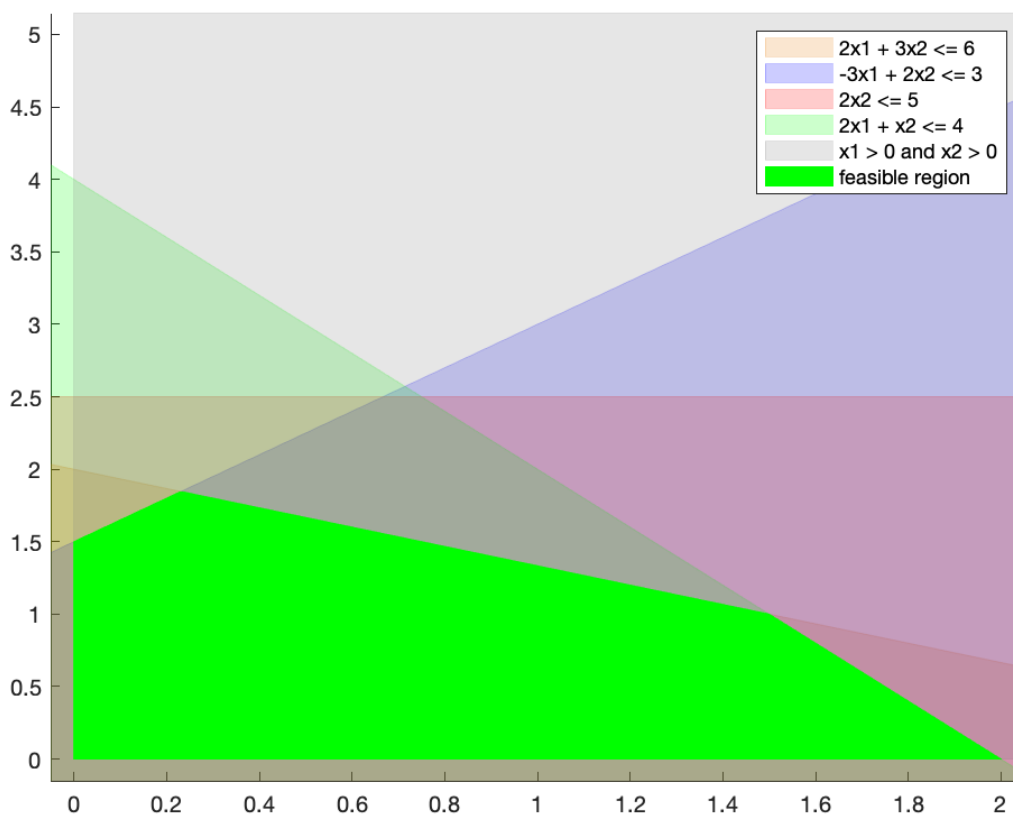

Figure 1: Plot of feasible region and constraints

Exercise 2.2

According to Nocedal, a vector x is a basic feasible point if it is in the feasible region and if there exists a subset β of the index set $1, 2, \ldots, n$ such that:

- β contains exactly m indices, where m is the number of rows of A;
- For any $i \notin \beta$, $x_i = 0$, meaning the bound $x_i \geq 0$ can be inactive only if $i \in \beta$;
- The m x m matrix B defined by $B = [A_i]_{i \in \beta}$ (where A_i is the i-th column of A) is non-singular, i.e. all columns corresponding to the indices in β are linearly independent from each other.

The geometric interpretation of basic feasible points is that all of them are verticies of the polytope that bounds the feasible region. We will use this proven property to manually solve the constrained minimization problem presented in this section by aiding us with the graphical plot of the feasible region in figure 1.

Exercise 2.3

Since the geometrical interpretation of the definition of basic feasible point states that these point are non-other than the vertices of the feasible region, we first look at the plot above and to these points (i.e. the verticies of the bright green non-trasparent region). Then, we look which constraint boundaries cross these edges, and we formulate an algebraic expression to find these points. In clockwise order, we have:

- The lower-left point at the origin, given by the boundaries of the constraints $x_1 \geq 0$ and $x_2 \geq 0$:

$$x_1^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- The top-left point, at the intersection of constraint boundaries $x_1 \geq 0$ and $-3x_1 + 2x_2 \leq 3$:

$$x_1 = 0 \quad 2x_2 = 3 \Leftrightarrow x_2 = \frac{3}{2} \quad x_2^* = \frac{1}{2} \cdot \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

- The top-center-left point at the intersection of constraint boundaries $-3x_1 + 2x_2 \leq 3$ and $2x_1 + 3x_2 \leq 6$:

$$-3x_1 + 2x_2 + 3x_1 + \frac{9}{2}x_2 = \frac{13}{2}x_2 = 3 + 9 = 12 \Leftrightarrow x_2 = 12 \cdot \frac{2}{13} = \frac{24}{13}$$

$$-3x_1 + 2 \cdot \frac{24}{13} = 3 \Leftrightarrow x_1 = \frac{39 - 48}{13} \cdot \frac{1}{-3} = \frac{3}{13} \quad x_3^* = \frac{1}{13} \cdot \begin{bmatrix} 3 \\ 24 \end{bmatrix}$$

- The top-center-right point at the intersection of constraint boundaries $2x_1 + 3x_2 \leq 6$ and $2x_1 + x_2 \leq 4$:

$$2x_1 + 3x_2 - 2x_1 - x_2 = 2x_2 = 6 - 4 = 2 \Leftrightarrow x_2 = 1 \quad 2x_1 + 1 = 4 \Leftrightarrow x_1 = \frac{3}{2} \quad x_4^* = \frac{1}{2} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

- The right point at the intersection of $2x_1 + x_2 \leq 4$ and $x_2 \geq 0$:

$$x_2 = 0 \quad 2x_1 + 0 = 4 \Leftrightarrow x_1 = 2 \quad x_5^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Therefore, x_1^* to x_5^* are all of the basic feasible points for this constrained minimization problem.

We then compute the objective function value for each basic feasible point found, The smallest objective value will correspond with the constrained minimizer problem solution.

$$x_1^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad f(x_1^*) = 4 \cdot 0 + 3 \cdot 0 = 0$$

$$x_2^* = \frac{1}{2} \cdot \begin{bmatrix} 0 \\ 3 \end{bmatrix} \quad f(x_2^*) = 4 \cdot 0 + 3 \cdot \frac{3}{2} = \frac{9}{2}$$

$$x_3^* = \frac{1}{13} \cdot \begin{bmatrix} 3 \\ 24 \end{bmatrix} \quad f(x_3^*) = 4 \cdot \frac{3}{13} + 3 \cdot \frac{24}{13} = \frac{84}{13}$$

$$x_4^* = \frac{1}{2} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad f(x_4^*) = 4 \cdot \frac{3}{2} + 3 \cdot 1 = 9$$

$$x_5^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad f(x_5^*) = 4 \cdot 2 + 1 \cdot 0 = 8$$

Therefore, $x^* = x_1^* = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ is the global constrained minimizer.

Exercise 3

Exercise 3.1

Exercise 3.2

The MATLAB code used to find the solution can be found under section 3.2 of the `main.m` script. The solution is:

$$x = \begin{bmatrix} 0.7692 \\ -2.2308 \\ 2.2308 \end{bmatrix} \quad \lambda = \begin{bmatrix} -10.3846 \\ 2.1538 \end{bmatrix}$$