Optimization methods – Homework 2

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1 Exercise 1

1.1 Implement the matrix A and the vector b, for the moment, without taking into consideration the boundary conditions. As you can see, the matrix A is not symmetric. Does an energy function of the problem exist? Consider $N = 4$ and show your answer, explaining why it can or cannot exist.

Answer is a energy function does not exist. Since A is not symmetric (even if it is pd), the minimizer used for the c.g. method (i.e. $\frac{1}{2}x^T Ax - b^T x$ won't work since $x^T Ax$ might be negative and thus the minimizer does not point to the solution of $Ax = b$ necessairly

1.2 Once the new matrix has been derived, write the energy function related to the new problem and the corresponding gradient and Hessian.

we already enforce x(1) = x(n) = 0, since b(1) = b(n) = 0 and thus A(1, :) * x = b(0) = 0 and same for n can be solved only for x(1) = x(n) = 0size(A, 1)

The objective is therefore $\phi(x) = (1/2)x^T \overline{A}x - b^x$ with a and b defined above, gradient is $= \overline{A}x - b$, hessian is $= \overline{A}$

1.3 Write the Conjugate Gradient algorithm in the pdf and implement it Matlab code in a function called `CGSolve`.

See page 112 (133 for pdf) for the algorithm implementation

The solution of this task can be found in Section 1.3 of the script `main.m`.

1.4 Solve the Poisson problem.

The solution of this task can be found in Section 1.4 of the script `main.m`.

1.5 Plot the value of energy function and the norm of the gradient (here, use semilogy) as functions of the iterations.

The solution of this task can be found in Section 1.5 of the script `main.m`.

1.6 Finally, explain why the Conjugate Gradient method is a Krylov subspace method.

Because theorem 5.3 holds, which itself holds mainly because of this (5.10, page 106 [127]):

$$r_{k+1} = r_k + a_k * A * p_k$$

2 Exercise 2

Consider the linear system $Ax = b$, where the matrix A is constructed in three different ways:

- $A =$ diag([1:10])
- $A =$ diag(ones(1,10))
- $A =$ diag([1, 1, 1, 3, 4, 5, 5, 5, 10, 10])
- $A =$ diag([1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0])

2.1 How many distinct eigenvalues has each matrix?

Each matrix has a distinct number of eigenvalues equal to the number of distinct elements on its diagonal. So, in order, each A has respectively 10, 1, 5, and 10 distinct eigenvalues.

2.2 Construct a right-hand side $b =$rand(10,1) and apply the Conjugate Gradient method to solve the system for each A.

The solution of this task can be found in section 2.2 of the `main.m` MATLAB script.

2.3 Compute the logarithm energy norm of the error for each matrix and plot it with respect to the number of iteration.

The solution of this task can be found in section 2.3 of the `main.m` MATLAB script.

2.4 Comment on the convergence of the method for the different matrices. What can you say observing the number of iterations obtained and the number of clusters of the eigenvalues of the related matrix?

The method converges quickly for each matrix. The fastest convergence surely happens for $A2$, which is the identity matrix and therefore makes the $Ax = b$ problem trivial.

For all the other matrices, we observe the energy norm of the error decreasing exponentially as the iterations increase, eventually reaching 0 for the cases where the method converges exactly (namely on matrices $A1$ and $A3$).

Other than for the fourth matrix, the number of iterations is exactly equal to the number of distinct eigenvalues for the matrix. That exception on the fourth matrix is simply due to the tolerance termination condition holding true for an earlier iteration, i.e. we terminate early since we find an approximation of x with residual norm below 10^{-8}.